System Cascade Connection

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Given two discrete-time systems A and B connected in cascade to form a new system $C = x \mapsto B(A(x))$.

1 Linearity

If A and B are linear, i.e. for all signals x_i and scalars a_i,

$$A\left(n \mapsto \sum_i a_i x_i[n]\right) = n \mapsto \sum_i a_i A(x_i)[n]$$
$$B\left(n \mapsto \sum_i a_i x_i[n]\right) = n \mapsto \sum_i a_i B(x_i)[n]$$

then C is also linear

$$\begin{aligned}
C\left(n \mapsto \sum_i a_i x_i[n]\right) &= B\left(A\left(n \mapsto \sum_i a_i x_i[n]\right)\right)\\
&= B\left(n \mapsto \sum_i a_i A(x_i)[n]\right)\\
&= n \mapsto \sum_i a_i B(A(x_i))[n]\\
&= n \mapsto \sum_i a_i C(x_i)[n]
\end{aligned}$$

2 Time Invariance

If A and B are time invariant, i.e. for all signals x and integers k,

$$A(n \mapsto x[n-k]) = n \mapsto A(x)[n-k]$$
$$B(n \mapsto x[n-k]) = n \mapsto B(x)[n-k]$$

then C is also time invariant

$$\begin{aligned} C(n \mapsto x[n-k]) &= B(A(n \mapsto x[n-k])) \\ &= B(n \mapsto A(x)[n-k]) \\ &= n \mapsto B(A(x))[n-k] \\ &= n \mapsto C(x)[n-k] \end{aligned}$$

3 LTI Ordering

If A and B are linear and time-invariant, there exists signals g and h such that for all signals x, $A = x \mapsto x * g$ and $B = x \mapsto x * h$, thus

$$B(A(x)) = B(x * g) = x * g * h = x * h * g = A(x * h) = A(B(x))$$

or interchanging A and B order does not change C.

4 Causality

If A and B are causal, i.e. for all signals x, y and integers k,

$$\begin{aligned} x[n] = y[n] \quad \forall n < k \implies &\begin{cases} A(x)[n] = A(y)[n] & \forall n < k \\ B(x)[n] = B(y)[n] & \forall n < k \end{cases} \\ \implies B(A(x))[n] = B(A(y))[n] \quad \forall n < k \iff &C(x)[n] = C(y)[n] \quad \forall n < k \end{aligned}$$

then C is also causal.

5 BIBO Stability

If A and B are stable, i.e. there exists a signal x and scalars a, b that

$$\begin{aligned} |x[n]| < a \quad \forall n \in Z &\implies |A(x)[n]| < b \quad \forall n \in Z \\ |x[n]| < a \quad \forall n \in Z &\implies |B(x)[n]| < b \quad \forall n \in Z \end{aligned}$$

then C is also stable, i.e. there exists a signal x and scalars a, b, c that

$$\begin{aligned} |x[n]| < a \, \forall n \in Z &\implies |A(x)[n]| < b \, \forall n \in Z \\ &\implies |B(A(x))[n]| < c \, \forall n \in Z \iff |C(x)[n]| < c \, \forall n \in Z \end{aligned}$$