Calculus Homework

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2 Limits

2.3 Limit Laws

Evaluate the limit:

$$\lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}} = \sqrt{\lim_{x\to 2}\frac{2x^2+1}{3x-2}} = \sqrt{\frac{2\cdot 2^2+1}{3\cdot 2-2}} = \frac{3}{2} \tag{9}$$

$$\lim_{x\to 4}\frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4}\frac{x}{x+1} = \frac{4}{5} \tag{12}$$

$$\begin{aligned}\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t} &= \lim_{t\to 0}\frac{2t}{t(\sqrt{1+t}+\sqrt{1-t})}\\ &= \lim_{t\to 0}\frac{2}{(\sqrt{1+t}+\sqrt{1-t})}\\ &= \frac{2}{\sqrt{1}+\sqrt{1}}\\ &= 1 \end{aligned} \tag{25}$$

40. Prove that $\lim_{x\to 0^+}\sqrt{x}e^{\sin\frac{\pi}{x}} = 0$.

Given $\varepsilon > 0$, let $\delta = \frac{1}{9}\varepsilon^2$. If $0 < x < 0 + \delta$ then

$$0 < \sqrt{x}e^{\sin\frac{\pi}{x}} \le e\sqrt{\delta} < 3\sqrt{\frac{\varepsilon^2}{9}} \implies |\sqrt{x}e^{\sin\frac{\pi}{x}} - 0| < \varepsilon$$

Thus, by the definition of right-hand limit,

$$\lim_{x\to 0^+}\sqrt{x}e^{\sin\frac{\pi}{x}} = 0$$

59. Prove that $\lim_{x\to 0} f(x) = 0$ if

$$f(x) = \begin{cases} x^2 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$

Given $\varepsilon > 0$, let $\delta = \sqrt{\varepsilon}$. If $0 < |x - 0| < \delta$, then $0 < x^2 < \varepsilon$ or $|f(x) - 0| < \varepsilon$. Thus, by the definition of a limit,

$$\lim_{x\to 0} f(x) = 0$$

61. If $f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$ and $g(x) = \begin{cases} 0 \text{ if } x \geq 0 \\ 1 \text{ if } x < 0 \end{cases}$ then $f(x)g(x) = 0$. Thus $\lim_{x\to 0} f(x)g(x) = 0$ though neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

2.4 The precise definition of a limit

3. Given $f(x) = \sqrt{x}$, if $|x - 4| < 1.44$ then $|\sqrt{x} - 2| < 0.4$.
21. Prove that $\lim_{x\to 2} \frac{x^2+x-6}{x-2} = 5$.

Given $\varepsilon > 0$, let $\delta = \varepsilon$. If $0 < |x - 2| < \delta$, then

$$|x + 3 - 5| < \varepsilon \iff \left|\frac{x^2 + x - 6}{x - 2} - 5\right| < \varepsilon$$

Thus, by the definition of a limit, $\lim_{x\to 2} \frac{x^2+x-6}{x-2} = 5$.
39. Prove that $\lim_{x\to 0} f(x)$ does not exist if

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational} \end{cases}$$

.

Suppose $\lim_{x\to 0} f(x) = L$, hence by the definition of limit, for every $\varepsilon > 0$, there exists $\delta > 0$ that

$$0 < |x - 0| < \delta \Rightarrow |f(x) - L| < \varepsilon \qquad (*)$$

For $L = 0$, consider $\varepsilon = |L - 1|$. For every δ, there is at least one irrational $x \in (0, \delta)$, which turns $(*)$ into a false statement:

$$0 < |x| < \delta \Rightarrow |1 - L| < |L - 1|$$

For $L \neq 0$, consider $\varepsilon = |L|$. For every δ, there is at least one rational $x \in (0, \delta)$, which turns $(*)$ into a false statement:

$$0 < |x| < \delta \Rightarrow |L| < |L|$$

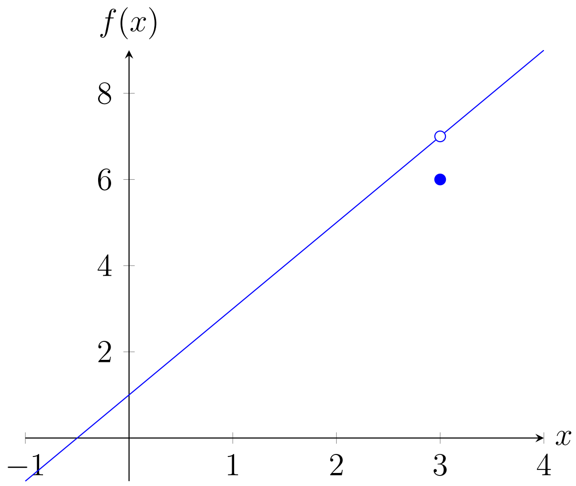
Conclusion: The assumption is incorrect; in other words, $\lim_{x\to 0} f(x)$ does not exist.

2.5 Continuity

22. Explain why the function f is discontinuous at the given number $a = 3$.

$$f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3} \text{ if } x \neq 3 \\ 6 \text{ if } x = 3 \end{cases}$$
$$= \begin{cases} 2x+1 \text{ if } x \neq 3 \\ 6 \text{ if } x = 3 \end{cases}$$

Since $\lim_{x\to 3} f(x) = \lim_{x\to 3}(2x+1) = 7 \neq 6 = f(3)$, f is discontinuous at 3.



26. $G(x) = \frac{x^2+1}{2x^2-x-1}$ is a rational function so it is continuous at every number in its domain.
38. Since *arctan* is an inverse trigonometric function and thus continuous at every number in its domain and $\lim_{x\to 2} \frac{x^2-4}{3x^2-6x} = \lim_{x\to 2} \frac{x+2}{3x} = \frac{2}{3}$,

$$\lim_{x\to 2} \arctan \frac{x^2-4}{3x^2-6x} = \arctan \frac{2}{3}$$

2.6 To Infinity and Beyond!

Find the limit:

$$\lim_{x\to -\infty} \frac{\sqrt{9x^6-x}}{x^3+1} = \lim_{x\to -\infty} \frac{\sqrt{9-\frac{1}{x^5}}}{-1-\frac{1}{x^3}} = -3 \tag{24}$$

2.7 Derivatives

24. If $g(x) = x^4 - 2$,

$$\begin{aligned} g'(1) &= \lim_{h\to 0} \frac{g(1+h) - g(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)^4 - 2 - (1^4 - 2)}{h} \\ &= \lim_{h\to 0} \frac{h^4 + 4h^3 + 6h^2 + 4h + 1 - 1}{h} \\ &= \lim_{h\to 0} (h^3 + 4h^2 + 6h + 4) \end{aligned}$$

An equation of the tangent line to g at $(1, -1)$:

$$y - g(1) = g'(1)(x - 1) \iff y = 4x - 5$$

Determine whether $f'(0)$ exists.

$$f(x) = \begin{cases} x \sin \frac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases} \tag{53}$$

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h \sin \frac{1}{h}}{h} \\ &= \lim_{h\to 0} \sin \frac{1}{h} && \text{(does not exist)} \end{aligned}$$

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases} \tag{54}$$

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} \\ &= \lim_{h\to 0} h \sin \frac{1}{h} \end{aligned}$$

Since $\forall h \neq 0, -|h| \leq h \sin \frac{1}{h} \leq |h|$ and $\lim_{h\to 0}(-|h|) = \lim_{h\to 0} |h| = 0$, according to the Squeeze Theorem, $f'(0) = 0$.

3 Differentiation

3.4 The chain rule

Find the derivative of the function.

$$y = \cos\sqrt{\sin(\tan \pi x)} \tag{45}$$

$$\begin{aligned}
\dot{y} &= -\sqrt{\sin(\tan \pi x)}' \cdot \sin\sqrt{\sin(\tan \pi x)} \\
&= \frac{\sin'(\tan \pi x) \cdot \sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \\
&= \frac{\tan' \pi x \cdot \cos(\tan \pi x) \cdot \sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \\
&= \frac{\pi \sec^2 \pi x \cdot \cos(\tan \pi x) \cdot \sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}
\end{aligned}$$

$$y = [x + (x + \sin^2 x)^3]^4 \tag{46}$$

$$\begin{aligned}
\dot{y} &= 4[x + (x + \sin^2 x)^3]'[x + (x + \sin^2 x)^3]^3 \\
&= 4[1 + 3(x + \sin^2 x)'(x + \sin^2 x)^2][x + (x + \sin^2 x)^3]^3 \\
&= 4[1 + 3(1 + \sin 2x)(x + \sin^2 x)^2][x + (x + \sin^2 x)^3]^3
\end{aligned}$$

3.7 Applications in Sciences

9. A rock is thrown vertically upward from the surface of Mars, its height after t seconds is $h = 15t - 1.86t^2$.

$$\frac{\mathrm{d}h}{\mathrm{d}t}(2) = (t \mapsto 15 - 3.72t)(2) = 7.56 \text{ (m/s)} \tag{a}$$

$$h = 25 \iff 15t - 1.86t^2 = 25 \iff t = \frac{375 \mp 25\sqrt{39}}{93} \tag{b}$$

So at $t \approx 2.35$ s or $t \approx 5.71$ the Rock's height is 25 m. Its velocity at this point is

$$v = (t \mapsto 15 - 3.72t)\left(\frac{375 \mp 25\sqrt{39}}{93}\right) = \pm 6.24 \text{ (m/s)}$$

10. A particle moves with position function

$$s = t^4 - 4t^3 - 20t^2 + 20t \qquad t \geq 0$$

$$v = 20 \iff \dot{s} = 20 \iff 4t^3 - 12t^2 - 40t + 20 = 20 \tag{a}$$

Since t is nonnegative, the particle has a velocity of 20 m/s at $t = 0$ and $t = 5$ s.

$$a = 0 \iff \dot{v} = 0 \iff 12t^2 - 24t - 40 = 0 \tag{b}$$

Since t is nonnegative, the acceleration is 0 at $t = \sqrt{\frac{13}{3}} - 1$ s. This is when the instantaneous speed of the particle ($|v|$) reaches its critical value.

21. The force F acting on a body with velocity v and mass $m = m_0 \Big/ \sqrt{1 - \frac{v^2}{c^2}}$ (where m_0 is the mass of the particle at rest and c is the speed of light) is the rate of change of momentum:

$$\begin{aligned}
F &= \frac{\mathrm{d}(mv)}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) \\
&= m_0 \frac{\mathrm{d}v}{\mathrm{d}t} \cdot \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) \\
&= m_0 a \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) \\
&= m_0 a \frac{\sqrt{1 - \frac{v^2}{c^2}} - v \frac{\mathrm{d}\sqrt{1 - v^2/c^2}}{\mathrm{d}v}}{1 - \frac{v^2}{c^2}} \\
&= m_0 a \frac{\sqrt{1 - \frac{v^2}{c^2}} - \frac{v}{2} \cdot \frac{-2v}{c^2\sqrt{1 - v^2/c^2}}}{1 - \frac{v^2}{c^2}} \\
&= m_0 a \frac{\sqrt{1 - \frac{v^2}{c^2}} + \frac{v^2}{c^2\sqrt{1 - v^2/c^2}}}{1 - \frac{v^2}{c^2}} \\
&= m_0 a \frac{1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \\
&= \frac{m_0 a}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}
\end{aligned}$$

30. The frequency of vibrations a vibrating violin string is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}} \qquad T \geq 0, \rho > 0$$

(a) The rate of change of the frequency with respect to

(i) The length: $\frac{\mathrm{d}f}{\mathrm{d}L} = \frac{-1}{L^2}\sqrt{\frac{T}{\rho}}$.

(ii) The tension: $\frac{\mathrm{d}f}{\mathrm{d}T} = \frac{1}{4L\sqrt{T\rho}}$.

(iii) The density: $\frac{\mathrm{d}f}{\mathrm{d}L} = \frac{-1}{L2}\sqrt{\frac{T}{\rho^3}}$.

(b) The pitch of a note gets higher when the string is shorter and lower when the tension or density is increased.

35. Applying the gas law

$$PV = nRT \iff T = \frac{PV}{nR}$$

The rate of change of temperature can be easily calculated via differentiation:

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{PV}{nR}\right) \\ &= \frac{1}{nR}\left(P\frac{\mathrm{d}V}{\mathrm{d}t} + V\frac{\mathrm{d}P}{\mathrm{d}t}\right) \\ &= \frac{8.0 \cdot 0.15 + 10 \cdot 0.10}{10 \cdot 0.0821} \\ &= \frac{1.2 + 1}{10 \cdot 0.0821} \\ &= \frac{2}{10 \cdot 0.0821} \\ &= 2 \text{ (K/s)}\end{aligned}$$

(In the calculation above, significant figures are taken into consideration.)

3.8 Exponential Growth and Decay

4. Let $P(t)$ be the bacteria count after t hours. As the bacteria culture grows with constant relative growth rate,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \Longrightarrow P(t) = P(0)e^{kt}$$

Since $P(2) = 400$ and $P(6) = 25600$,

$$\begin{cases} P(0)e^{2k} = 400 \\ P(0)e^{6k} = 25600 \end{cases} \iff \begin{cases} P(0)e^{2k} = 400 \\ e^{4k} = 64 \end{cases} \iff \begin{cases} P(0)e^{2k} = 400 \\ e^{2k} = 8 \end{cases} \iff \begin{cases} P(0) = 50 \\ k = \frac{\ln 8}{2} \approx 104\% \end{cases}$$

Thus (a) the relative growth rate is 104%, (b) the initial size of the culture is 50 and (c) the number of bacteria after t hours is $50\sqrt{8^t}$.

The number of cells after 4.5 hours:

$$P(4.5) = 50\sqrt{8^{4.5}} \approx 5382 \qquad \text{(d)}$$

The rate of growth after 4.5 hours:

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t}(4.5) &= 50\frac{\mathrm{d}\sqrt{8}^t}{\mathrm{d}t}(4.5) \\ &= 50\left(t \mapsto \sqrt{8^t}\ln\sqrt{8}\right)(4.5) \\ &= 25 \cdot 8^{2.25}\ln 8 \\ &\approx 5596 \text{ (bacteria per minute)} \end{aligned} \qquad \text{(e)}$$

The population reach 50000 when

$$50\sqrt{8^t} = 50000 \iff 8^t = 10^6 \iff t = \log_2 100 \approx 6.64 \text{ (days)} \qquad \text{(f)}$$

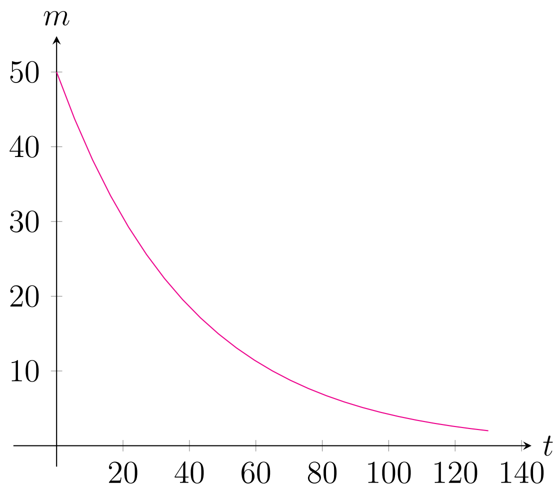
8. Given 50 mg of ^{90}Sr which has a half-life of 28 days.

(a) Formula of the mass remaining after t days: $m(t) = 50 \cdot 2^{-t/28}$.

(b) The mass remaining after 40 days: $m(40) = 50 \cdot \frac{1}{2}^{10/7} \approx 19$ (mg).

(c) To decay to a mass of 2 mg, it takes $-28\log_2 \frac{2}{50} \approx 130$ (days).

(d) The graph of the mass function:



16. Let $T(t)$ be the temperature of the coffee after t minutes. The surrounding temperature is 20°C, so Newton's Law of Cooling states that

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - 20)$$

If we let $y = T - 20$, then $y(0) = T(0) - 20 = 95 - 20 = 75$, so y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky \iff y(t) = 75e^{kt}$$

When the temperature of the coffee is 70°C, its cooling rate is 1°C per minute, i.e.

$$\begin{cases} y(t) + 20 = 70 \\ ky(t) = -1 \end{cases} \iff \begin{cases} y(t) = 50 \\ k = \frac{-1}{50} \end{cases}$$
$$\implies 75e^{-t/50} = 50 \iff t = 50 \ln 1.5 \approx 20 \text{ (minutes)}$$

3.9 Related rates

10. A particle is moving along a hyperbola $xy = 8$

$$\begin{aligned} \implies \frac{\mathrm{d}(xy)}{\mathrm{d}t} = \frac{\mathrm{d}8}{\mathrm{d}t} &\iff y\frac{\mathrm{d}x}{\mathrm{d}t} + x\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \\ &\iff 2 \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + 4 \cdot 3 = 0 \\ &\iff \frac{\mathrm{d}x}{\mathrm{d}t} = -6 \text{ (cm/s)} \end{aligned}$$

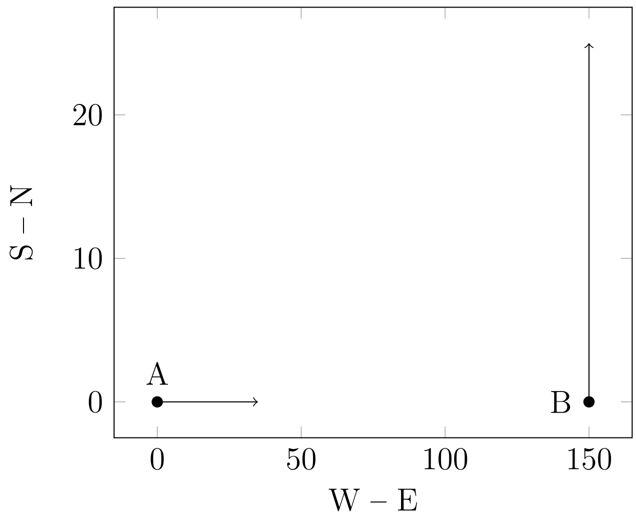
12. Let $D(t)$ (cm) be the diameter of the ball at minute t, its surface area is $A(D) = \pi D^2$ (cm^2).

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 1 \iff \frac{\mathrm{d}A}{\mathrm{d}D} \cdot \frac{\mathrm{d}D}{\mathrm{d}t} = 1 \iff 2\pi D \frac{\mathrm{d}D}{\mathrm{d}t} = 1 \iff \frac{\mathrm{d}D}{\mathrm{d}t} = \frac{1}{2\pi D}$$

Thus the decreasing rate of the diameter when it is 10 cm:

$$\frac{\mathrm{d}D}{\mathrm{d}t}(10) = \frac{1}{20\pi} \text{ (cm/s)}$$

14.



$$\begin{cases} \Delta x(t) = x_B(t) - x_A(t) \\ \Delta y(t) = y_B(t) - y_A(t) \end{cases} \iff \begin{cases} \Delta x(t) = 150 - 35t \\ \Delta y(t) = 25t \end{cases}$$

$$\implies \Delta s(t) = \sqrt{1850t^2 - 10500t + 22500} \implies \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1850t - 5250}{\sqrt{1850t^2 - 10500t + 22500}}$$

$$\implies \frac{\mathrm{d}s}{\mathrm{d}t}(4) = \frac{1850 \cdot 4 - 5250}{\sqrt{1850 \cdot 16 - 10500 \cdot 4 + 22500}} = \frac{2150}{\sqrt{10100}} = \frac{215\sqrt{101}}{101} \approx 21 \text{ (km/h)}$$

27. Let $h(t)$ (ft) be the height of the cone at minute t. Volume of the cone is

$$V(h) = \frac{\pi h^2}{12} \implies \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi h}{6} \cdot \frac{\mathrm{d}h}{\mathrm{d}t} = 30 \iff \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{180}{\pi h}$$

$$\implies \frac{\mathrm{d}h}{\mathrm{d}t}(10) = \frac{180}{10\pi} = \frac{18}{\pi} \approx 5.7 \text{ (ft/s)}$$

4 Applications of derivative

4.1 Max and Min

Find the absolute min and max values of f.

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \qquad [-2, 3] \tag{51}$$

Since $f'(x) = 12x^3 - 12x^2 - 24x$, we have $f'(x) = 0$ when $x \in \{-1, 0, 2\}$. The values of f at these critical numbers are

$$\begin{aligned} f(-1) &= 3 + 4 - 12 + 1 = -4 \\ f(0) &= 1 \\ f(2) &= 48 - 32 - 48 + 1 = -31 \end{aligned}$$

The values of f at endpoints are

$$\begin{aligned} f(-2) &= 48 + 32 - 48 + 1 = 33 \\ f(3) &= 243 - 108 - 108 + 1 = 28 \end{aligned}$$

Comparing these five numbers and using the Closed Interval Method, we see that the absolute minimum value is $f(2) = -31$ and the absolute maximum value is $f(-2) = 33$.

$$f(t) = t\sqrt{4 - t^2}, \qquad [-1, 2] \tag{55}$$

Since $f'(t) = \frac{4-2t^2}{\sqrt{4-t^2}}$, with $t \in [-1, 2]$, we have $f'(t) = 0$ when $t = \sqrt{2}$. The value of f at this critical number is $f(\sqrt{2}) = 2$. The value of f at endpoints are $f(-1) = -\sqrt{3}$ and $f(2) = 0$. Comparing these 3 numbers and using the Closed Interval Method, we see that the absolute minimum value is $f(-1) = -\sqrt{3}$ and the absolute maximum value is $f(\sqrt{2}) = 2$.

$$f(x) = xe^{-x^2/8}, \qquad [-1, 4] \tag{59}$$

With $x \in [-1, 4]$, we have $f'(x) = \left(1 - \frac{x^2}{4}\right) e^{-x^2/8}$ when $x = 2$. Comparing values of f at this critical number and endpoints, the minimum value is $f(-1) = -e^{-1/8}$ and the maximum value is $f(2) = 2e^{-1/2}$.

4.2 The Mean Theorem

26. Let h be the function that $h(x) = f(x) - g(x)$. Since both f and g are continuous on $[a, b]$ and differentiable on (a, b), h inherits the same properties.

By applying the Mean Value Theorem to h on the interval $[a, b]$, we get a number $c \in (a, b)$ such that

$$\begin{aligned} & h(b) - h(a) = (b-a)h'(c) \\ \iff & f(b) - g(b) - f(a) + g(a) = (b-a)(f'(c) - g'(c)) \\ \iff & f(b) - g(b) = (b-a)(f'(c) - g'(c)) \end{aligned}$$

$b - a > 0$ and $f'(c) - g'(c) < 0$ so $f(b) - g(b) < 0$ or $f(b) < g(b)$.

$$\begin{aligned} x > 0 &\iff x + 1 > 1 \\ &\iff \sqrt{x+1} > 1 \\ &\iff \sqrt{x+1} - 1 > 0 \\ &\implies \left(\sqrt{x+1} - 1\right)^2 > 0 \\ &\iff x + 1 - 2\sqrt{x+1} + 1 > 0 \\ &\iff x + 2 > 2\sqrt{x+1} \\ &\iff \sqrt{1+x} < 1 + \frac{1}{2}x \end{aligned} \tag{27}$$

33. Prove the identity

$$\arcsin \frac{x-1}{x+1} = 2 \arctan \sqrt{x} - \frac{\pi}{2}$$

Let $\frac{-\pi}{2} \leq y = \arcsin \frac{x-1}{x+1} \leq \frac{\pi}{2}$ and $z = \arctan \sqrt{x}$, then

$$\begin{aligned} \begin{cases} \sin y = \frac{x-1}{x+1} \\ \tan z = \sqrt{x} \end{cases} &\implies \begin{cases} \frac{\mathrm{d} \sin y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{x-1}{x+1}\right) \\ \frac{\mathrm{d} \tan z}{\mathrm{d}x} = \frac{\mathrm{d}\sqrt{x}}{\mathrm{d}x} \end{cases} \\ &\iff \begin{cases} \cos y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{(x+1)^2} \\ (\tan^2 z + 1) \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} \end{cases} \\ &\iff \begin{cases} \sqrt{1 - \sin^2 y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{(x+1)^2} \\ \left(\sqrt{x}^2 + 1\right) \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} \end{cases} \\ &\iff \begin{cases} \sqrt{\frac{4x}{(x+1)^2}} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{(x+1)^2} \\ (x+1) \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} \end{cases} \\ &\iff \begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{|x+1|\sqrt{x}} \\ \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{2(x+1)\sqrt{x}} \end{cases} \end{aligned}$$

For all $x \geq 0$ or $x + 1 \geq 1 > 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1}\right) &= 2\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x}\\ &= \frac{2}{2(x+1)\sqrt{x}} - \frac{1}{(x+1)\sqrt{x}}\\ &= 0\end{aligned}$$

Thus the function $x \mapsto 2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1}$ is constant on its domain $[0, \infty)$. Consequently, in $[0, \infty)$

$$\begin{aligned}2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1} &= \left(x \mapsto 2\arctan\sqrt{x} - \arcsin\frac{x-1}{x+1}\right)(0)\\ &= 2\arctan\sqrt{0} - \arcsin\frac{-1}{1}\\ &= 0 - \frac{-\pi}{2}\\ &= \frac{\pi}{2}\\ \iff \arcsin\frac{x-1}{x+1} &= 2\arctan\sqrt{x} - \frac{\pi}{2}\end{aligned}$$

4.3 Shape of a graph

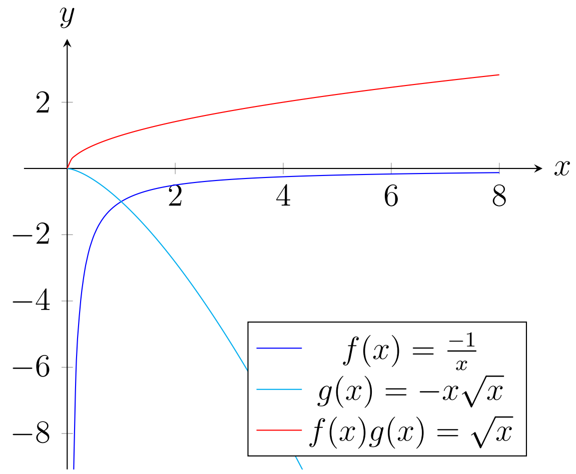
75. Given two funtions f and g which are positive and concave upward on I, i.e. for all x in I

$$\begin{cases} f(x) > 0\\ f''(x) > 0\\ g(x) > 0\\ g''(x) > 0\end{cases}$$

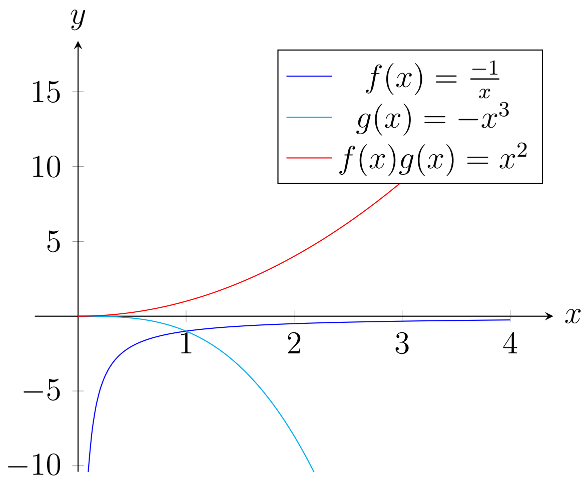
Second derivative of the product function fg:

$$(f(x)g(x))'' = (f'(x)g(x) + f(x)g'(x))' = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$$

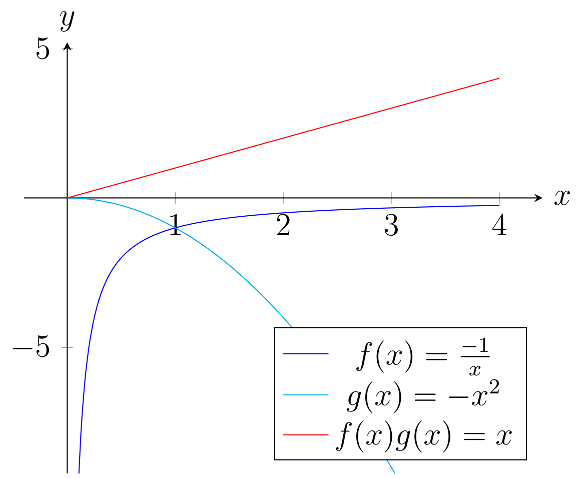
If f and g are both increasing or decreasing, then $f'(x)g'(x) > 0$, which means $\forall x \in I, (f(x)g(x))'' > 0$, or fg is concave upward on I. Otherwise, f is increasing and g is decreasing for instance, fg may be either concave upward, concave downward or linear:



y
x
2
−2
−4
−6
−8
2
4
6
8
$f(x) = \frac{-1}{x}$
$g(x) = -x\sqrt{x}$
$f(x)g(x) = \sqrt{x}$



y
x
15
10
5
−5
−10
1
2
3
4
$f(x) = \frac{-1}{x}$
$g(x) = -x^3$
$f(x)g(x) = x^2$



y
x
5
−5
1
2
3
4
$f(x) = \frac{-1}{x}$
$g(x) = -x^2$
$f(x)g(x) = x$

76. In order for $h = f(g(x))$ to be concave upward on $\mathbb{R}$

$$\begin{aligned} h'' > 0 &\iff (f \circ g)'' > 0 \\ &\iff ((f' \circ g) \cdot g')' > 0 \\ &\iff (f' \circ g)' \cdot g' + (f' \circ g) \cdot g'' > 0 \\ &\iff (f'' \circ g) \cdot (g')^2 + (f' \circ g) \cdot g'' > 0 \end{aligned}$$

Because f and g are given to be concave upward on $\mathbb{R}$, i.e. $f'' > 0$ and $g'' > 0$, and $\forall x \in \mathbb{R}, g^2(x) \geq 0$, so if $f' > 0$ or f is an increasing function, h will be concave upward.

77. Show that $\tan x > x$ for $0 < x < \frac{\pi}{2}$.

Let f be the function that $f(x) = \tan x - x$. On $(0, \frac{\pi}{2})$, $\sin x \cos x \neq 0$ thus $\tan x$ exists and is nonzero. Therefore, $f'(x) = \tan^2(x) > 0$ or f is increasing on $[0, \frac{\pi}{2}]$, which means for all x in $(0, \frac{\pi}{2})$,

$$f(x) > f(0) \iff \tan x - x > \tan 0 - 0 \iff \tan x > x$$

78. Use mathematical induction to prove that for all positive integer n,

$$\forall x \geq 0, \qquad e^x \geq 1 + \sum_{i=1}^{n} \frac{x^i}{i!} \qquad (*)$$

Let f be the function of domain $[0, \infty)$ that $f(x) = e^x - x$, then for all $x \geq 0$, $f'(x) = e^x - 1 > f'(0) = 0$ (since it is obvious that f' is an increasing function). Hence $\forall x \geq 0, e^x - x > e^0 - 0 = 1 \iff \forall x \geq 0, e^x > 1 + x$, i.e. $(*)$ is true for $n = 1$.

Suppose that $(*)$ is also true for $n = k$ ($k \in \mathbb{N}^*$). For all nonnegative x,

$$e^x \geq 1 + \sum_{i=1}^{k} \frac{x^i}{i!} \iff e^x - 1 - \sum_{i=1}^{k} \frac{x^i}{i!} \geq 0$$

Let g be the function of domain $[0, \infty)$ that $g(x) = e^x - \sum_{i=1}^{k+1} \frac{x^i}{i!}$, then for all positive x

$$g'(x) = e^x - \sum_{i=1}^{k+1} \frac{ix^{i-1}}{i!} = e^x - 1 - \sum_{i=2}^{k+1} \frac{x^{i-1}}{(i-1)!} = e^x - 1 - \sum_{i=1}^{k} \frac{x^i}{i!} \geq 0$$

This means g in a non-decreasing function on $[0, \infty)$

$$e^x - \sum_{i=1}^{k+1} \frac{x^i}{i!} \geq e^0 - \sum_{i=1}^{k+1} \frac{0^i}{i!} = 1$$

This expression shows that $(*)$ is true for $n = k + 1$. Therefore, by mathematical induction, it is true for all positive integers n.

4.4 Rule of the Hospital

Find the limit
54. Since $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\sqrt{x}\cdot\ln x} = e^{\lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}}} = e^{\lim_{x\to 0^+} \frac{-2x\sqrt{x}}{x}} = e^{-2\lim_{x\to 0^+}\sqrt{x}} = e^0 = 1$$

60. Since $\lim_{x\to\infty} \ln 2 \ln x = \lim_{x\to\infty}(1+\ln x) = \infty$

$$\lim_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}} = e^{\lim_{x\to\infty} \frac{\ln 2 \ln x}{1+\ln x}} = e^{\lim_{x\to\infty} \ln 2} = e^{\ln 2} = 2$$

4.7 Optimization Problems

44. Given $E(v) = \frac{aLv^3}{v-u}$

$$\frac{\mathrm{d}E}{\mathrm{d}v} = aL\frac{3v^2(v-u)-v^3}{(v-u)^2} = aL\frac{2v^3-3uv^2}{(v-u)^2}$$

Since $v > u > 0$, E has only one absolute extreme value, at the only critical number $v = 1.5u$. Applying the First Derivative Test for Absolute Extreme Values, $v = 1.5u$ is shown to be the value of v that minimizes E.
45. Given $S = 6sh + \frac{3}{2}s^2(\sqrt{3}\cdot\csc\theta - \cot\theta)$.

$$\frac{\mathrm{d}S}{\mathrm{d}\theta} = \frac{3}{2}s^2\frac{\mathrm{d}}{\mathrm{d}\theta}(\sqrt{3}\cdot\csc\theta - \cot\theta) = \frac{3s^2(1-\sqrt{3}\cdot\cos\theta)}{2\sin^2\theta}$$

We have $\frac{\mathrm{d}S}{\mathrm{d}\theta} = 0$ when $\theta = \arccos\frac{\sqrt{3}}{3}$. Applying the First Derivative Test for Absolute Extreme Values, this value of θ is shown to minimize S to $6sh + \frac{3s^2}{\sqrt{2}}$.
76. Using Poiseuille's Law, we have the total resistance of the blood along the path ABC is

$$R = R_{AB} + R_{BC} = C\frac{a-b\cot\theta}{r_1^4} + C\frac{b}{r_2^4\sin\theta} = C\left(\frac{a-b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$
$$\implies \frac{\mathrm{d}R}{\mathrm{d}\theta} = \frac{Cb}{r_1^4\sin^2\theta} - \frac{Cb\cos\theta}{r_2^4\sin^2\theta} = \frac{Cb}{sin^2\theta}\left(\frac{1}{r_1^4} - \frac{\cos\theta}{r_2^4}\right)$$

We have $\frac{\mathrm{d}R}{\mathrm{d}\theta} = 0$ when $\cos\theta = (r_2/r_1)^4$. At this angle, the resistance is minimized (can be shown using the First Derivative Test for Absolute Extreme Values, but like in the two previous exercises, I'm too lazy to evaluate it). When $\frac{r_2}{r_1} = \frac{2}{3}$, the optimal branching angle is $\theta \approx 79°$.

5 Integral

5.1 Areas

4. Estimate the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{4-0}{n} \sum_{i=1}^{n} \sqrt{\frac{4i}{n}} = \lim_{n\to\infty} \frac{8}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}}$$

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{4-0}{n} \sum_{i=0}^{n-1} \sqrt{\frac{4i}{n}} = \lim_{n\to\infty} \frac{8}{n} \sum_{i=0}^{n-1} \sqrt{\frac{i}{n}}$$

For estimation, consider $n \to 4$:

$$\lim_{n\to 4} R_n = \frac{8}{4} \sum_{i=1}^{4} \sqrt{\frac{i}{4}} = \sum_{i=1}^{4} \sqrt{i} = 1 + \sqrt{2} + \sqrt{3} + 2 \approx 6.1463$$

$$\lim_{n\to 4} L_n = \frac{8}{4} \sum_{i=0}^{3} \sqrt{\frac{i}{4}} = \sum_{i=0}^{3} \sqrt{i} = 0 + 1 + \sqrt{2} + \sqrt{3} \approx 4.1463$$

5. Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2+1}{n} \sum_{i=1}^{n} f\left(-1 + i\frac{2+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(1 + \left(\frac{3i}{n} - 1\right)^2\right)$$

Similarly,

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{3}{n} \sum_{i=0}^{n-1} \left(1 + \left(\frac{3i}{n} - 1\right)^2\right)$$

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(1 + \left(\frac{3i - 3/2}{n} - 1\right)^2\right)$$

For $n \to 3$,

$$\lim_{n\to 3} R_n = \sum_{i=1}^{3} \left(1 + (i-1)^2\right) = 1 + 2 + 5 = 8$$

$$\lim_{n\to 3} M_n = \sum_{i=1}^{3} \left(1 + \left(i - \frac{3}{2}\right)^2\right) = \frac{5}{4} + \frac{5}{4} + \frac{13}{4} = 5.75$$

$$\lim_{n\to 3} L_n = \sum_{i=0}^{2} \left(1 + (i-1)^2\right) = 2 + 1 + 2 = 5$$

For $n \to 6$,

$$\lim_{n\to 6} R_n = \frac{1}{2}\sum_{i=1}^{6}\left(1+\left(\frac{i}{2}-1\right)^2\right) = \frac{1}{2}\left(\frac{5}{4}+1+\frac{5}{4}+2+\frac{13}{4}+5\right) = 6.875$$
$$\lim_{n\to 6} M_n = \frac{1}{2}\sum_{i=1}^{6}\left(1+\left(\frac{2i-5}{4}\right)^2\right) = \frac{1}{2}\left(\frac{25}{16}+\frac{17}{16}+\frac{17}{16}+\frac{25}{16}+\frac{41}{16}+\frac{65}{16}\right) = 5.9375$$
$$\lim_{n\to 6} L_n = \frac{1}{2}\sum_{i=0}^{5}\left(1+\left(\frac{i}{2}-1\right)^2\right) = \frac{1}{2}\left(2+\frac{5}{4}+1+\frac{5}{4}+2+\frac{13}{4}\right) = 5.375$$

16. The height (in feet) above the earth's surface of the *Endeavour*, 62 seconds after liftoff, can be estimated with the assist of Python (which, coincidentally, has been utilized by NASA recently):

```
>>> time = 0, 10, 15, 20, 32, 59, 62, 125
>>> velocity = 0, 185, 319, 447, 742, 1325, 1445, 4151
>>> sum(map(int.__mul__, velocity,
...         map(int.__sub__, time[1:], time[:-1])))
122928
```

5.2 The Definite Integral

Evaluate the integral.

$$\int_2^5 (4-2x)\,\mathrm{d}x = 4x\big]_2^5 - x^2\big]_2^5 = 12 - 21 = -9 \tag{21}$$
$$\int_0^2 (2x-x^3)\,\mathrm{d}x = x^3\big]_0^2 - \frac{x^4}{4}\bigg]_0^2 = 9 - 16 = -7 \tag{24}$$

33. Evaluate integral by interpreting it in terms of areas.

$$\int_0^2 f(x)\,\mathrm{d}x = 4 \tag{a}$$
$$\int_0^5 f(x)\,\mathrm{d}x = 10 \tag{b}$$
$$\int_5^7 f(x)\,\mathrm{d}x = -3 \tag{c}$$
$$\int_0^9 f(x)\,\mathrm{d}x = \int_0^5 f(x)\,\mathrm{d}x + \int_5^9 f(x)\,\mathrm{d}x = 10 - 8 = 2 \tag{d}$$

50. Given $f(x) = \begin{cases} 3 \text{ for } x < 3 \\ x \text{ for } x \geq 3 \end{cases}$

$$\int_0^5 f(x)\,\mathrm{d}x = \int_0^3 3\,\mathrm{d}x + \int_3^5 x\,\mathrm{d}x = 3x]_0^3 + \frac{x^2}{2}\Big]_3^5 = 9 + 8 = 17$$

5.3 The Fundamental Theorem of Calculus

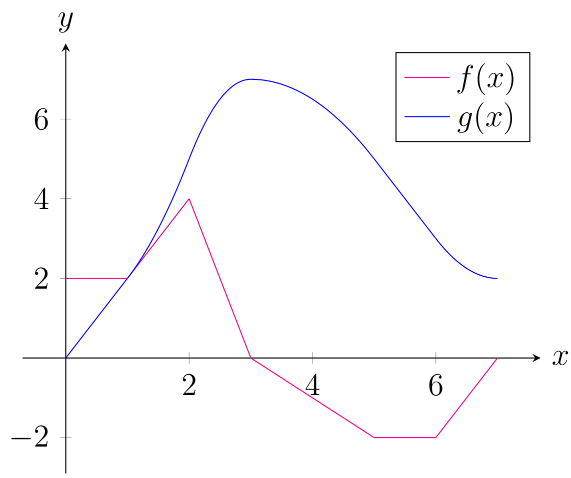
3. Let $g(x) = \int_0^x f(t)\,\mathrm{d}t$.

(a) By interpreting the above integral in terms of areas, we get $g(0) = 0$, $g(1) = 2$, $g(2) = 5$, $g(3) = 7$ and $g(6) = 3$.

(b) g is increasing on $(0, 3)$.

(c) g has a maximum value of 7 at $x = 3$.

(d)



Find the derivative of the function.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\,\mathrm{d}z &= \frac{\mathrm{d}}{\mathrm{d}\sqrt{x}}\left(\int_1^{\sqrt{x}} \frac{z^2}{z^4+1} dz\right)\frac{\mathrm{d}\sqrt{x}}{\mathrm{d}x} \\ &= \frac{x}{x^2+1}\cdot\frac{1}{2\sqrt{x}} \\ &= \frac{\sqrt{x}}{2x^2+2} \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\tan x} \sqrt{t+\sqrt{t}}\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}\tan x}\left(\int_0^{\tan x} \sqrt{t+\sqrt{t}}\,\mathrm{d}t\right)\frac{\mathrm{d}\tan x}{\mathrm{d}x} \\ &= \frac{\sqrt{\tan x + \sqrt{\tan x}}}{\cos^2 x} \end{aligned} \tag{15}$$

64. Given the **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t \Longrightarrow \operatorname{erf}'(x) = \frac{2e^{-x^2}}{\sqrt{\pi}}$$

$$\int_a^b e^{-t^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2} \int_a^b \operatorname{erf}'(t)\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}[\operatorname{erf}(b) - \operatorname{erf}(a)] \qquad \text{(a)}$$

With $y = e^{x^2} \operatorname{erf}(x)$

$$\begin{aligned} y' &= \left(e^{x^2}\right)' \operatorname{erf}(x) + e^{x^2} \operatorname{erf}'(x) \\ &= 2xe^{x^2} \operatorname{erf}(x) + e^{x^2} \frac{2e^{-x^2}}{\sqrt{\pi}} \\ &= 2xe^{x^2} \operatorname{erf}(x) + \frac{2}{\sqrt{\pi}} \\ &= 2xy + \frac{2}{\sqrt{\pi}} \qquad \text{(b)} \end{aligned}$$

5.4 Infinite Integral

56. Let $y(x)$ be the vertical postion at a distance of x miles from the start of the trail, then $y'(x) = f(x)$. Thus, $\int_3^5 f(x)\,\mathrm{d}x = y(5) - y(3)$, which is the vertical displacement from 3 to 5 miles.

63. Total mass of the rod:

$$\int_0^4 (9 + 2\sqrt{x})\,\mathrm{d}x = 9x]_0^4 + \left.\frac{2\sqrt{x^3}}{3}\right]_0^4 = 36 + \frac{16}{3} = 41\frac{1}{3} \text{ (kg)}$$

64. Amount of water flowing from the tank during the first 10 minutes:

$$\int_0^{10} (200 - 4t)\,\mathrm{d}t = 200t]_0^{10} - 2t^2\Big]_0^{10} = 2000 - 200 = 1800 \text{ (l)}$$

5.5 The Substitution Rule

74. Given $f(x) = \sin \sqrt[3]{x}$.

Since $f(-x) = \sin \sqrt[3]{-x} = \sin - \sqrt[3]{x} = -\sin \sqrt[3]{x} = -f(x)$, f is an odd function. Hence $\int_{-2}^3 f(x)\,\mathrm{d}x = \int_2^3 f(x)\,\mathrm{d}x$.

For $2 \le x \le 3$, $0 \le \sqrt[3]{2} \le \sqrt[3]{x} \le \sqrt[3]{3}\pi$, thus $\sin \sqrt[3]{x} \ge 0$ and $\int_2^3 f(x)\,\mathrm{d}x \ge 0$. Futhermore, $\sin \sqrt[3]{x} \le 1$ so $\int_2^3 f(x)\,\mathrm{d}x \le \int_2^3 \mathrm{d}x = 1$.

Evaluate the integral.

$$\begin{aligned}\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,\mathrm{d}x &= \int_{-2}^{2} x\sqrt{4-x^2}\,\mathrm{d}x + 3\int_{-2}^{2}\sqrt{4-x^2}\,\mathrm{d}x\\ &= 0+3\cdot 2\pi\\ &= 6\pi \end{aligned} \tag{77}$$

$$\begin{aligned}\int_{0}^{24}\left(85-0.18\cos\frac{\pi t}{12}\right)\,\mathrm{d}t &= 85t]_0^{24} - \frac{54}{25\pi}\int_0^{24}\frac{\pi t}{12}'\cos\frac{\pi t}{12}\,\mathrm{d}t\\ &= 2040 - \frac{54}{25\pi}\int_0^{2\pi}\cos x\,\mathrm{d}x\\ &= 2040 - \frac{54\sin x}{25\pi}\bigg]_0^{2\pi}\\ &= 2040 \end{aligned} \tag{80}$$

$$\begin{aligned}400+\int_0^3 450.268e^{1.12567t}\,\mathrm{d}t &= 400+400\int_0^3 1.12567e^{1.12567t}\,\mathrm{d}t\\ &= 400+\,400e^{1.12567t}\big]_0^3\\ &= 400e^{1.12567\cdot 3}\\ &\approx 11713 \end{aligned} \tag{82}$$

6 Applications of Integration

6.1 Areas Between Curves

Evaluate the integral

$$\begin{aligned}\int_{-1}^{1}\left|e^x-x^2+1\right|\,\mathrm{d}x &= \int_{-1}^{1}\left(e^x-x^2+1\right)\,\mathrm{d}x\\ &= \left[e^x-\frac{x^3}{3}+x\right]_{-1}^{1}\\ &= e-\frac{1}{3}+1-\frac{1}{e}-\frac{1}{3}+1\\ &= e-\frac{1}{e}+\frac{4}{3} \end{aligned} \tag{5}$$

$$\begin{aligned}\int_1^4 \left|x^2-3x+4\right| \,\mathrm{d}x &= \int_1^4 \left(x^2-3x+4\right) \,\mathrm{d}x \\ &= \left[\frac{x^3}{3}-\frac{3x^2}{2}+4x\right]_1^4 \\ &= \frac{64-1}{3}-\frac{48-3}{2}+16-4 \\ &= \frac{21}{2} \end{aligned} \tag{7}$$

$$\begin{aligned}\int_1^2 \left|\frac{1}{x}-\frac{1}{x^2}\right| \,\mathrm{d}x &= \int_1^2 \left(\frac{1}{x}-\frac{1}{x^2}\right) \,\mathrm{d}x \\ &= \left[\frac{1}{3x^3}-\frac{1}{2x^2}\right]_1^2 \\ &= \frac{1}{24}-\frac{1}{8}-\frac{1}{3}+\frac{1}{2} \\ &= \frac{1}{12} \end{aligned} \tag{9}$$

53. Find the values of c such that

$$\begin{aligned}\int_{-|c|}^{|c|} \left|x^2-c^2-c^2+x^2\right| \,\mathrm{d}x = 576 &\iff \int_{-|c|}^{|c|} \left(c^2-x^2\right) \,\mathrm{d}x = 288 \\ &\iff \left[c^2x-\frac{x^3}{3}\right]_{-|c|}^{|c|} = 288 \\ &\iff \frac{4\left|c^3\right|}{3} = 288 \\ &\iff |c| = 6 \\ &\iff c = \pm 6 \end{aligned}$$

54. Find the area of the region enclosed by the line $y = mx$ and the curve $y = \frac{x}{x^2+1}$.

Those two curves enclose a region if and only if the following equations has two unique solutions, i.e. $m \in (0,1)$

$$mx = \frac{x}{x^2+1} \iff mx^3+(m-1)x = 0 \iff x \in \left\{0, \pm\frac{1-m}{m}\right\}$$

The area of the region would then be

$$
\begin{aligned}
A &= \int_{\frac{m-1}{m}}^{\frac{1-m}{m}} \left| mx - \frac{x}{x^2+1} \right| \,\mathrm{d}x \\
&= \int_{\frac{m-1}{m}}^{0} \left(\frac{x}{x^2+1} - mx \right) \mathrm{d}x + \int_{0}^{\frac{1-m}{m}} \left(mx - \frac{x}{x^2+1} \right) \mathrm{d}x \\
&= \int_{\frac{m-1}{m}}^{0} \left(\frac{x}{x^2+1} - mx \right) \mathrm{d}x + \int_{0}^{\frac{1-m}{m}} \left(mx - \frac{x}{x^2+1} \right) \mathrm{d}x \\
&= \int_{\frac{m-1}{m}}^{0} \frac{1}{x^2+1} \cdot \frac{\mathrm{d}x^2}{\mathrm{d}x} \,\mathrm{d}x - \left.\frac{mx^2}{2}\right]_{\frac{m-1}{m}}^{0} + \left.\frac{mx^2}{2}\right]_{0}^{\frac{1-m}{m}} - \int_{0}^{\frac{1-m}{m}} \frac{1}{x^2+1} \cdot \frac{\mathrm{d}x^2}{\mathrm{d}x} \,\mathrm{d}x \\
&= \frac{(m-1)^2}{m} + 2 \int_{\left(\frac{m-1}{m}\right)^2}^{0} \frac{1}{x+1} \,\mathrm{d}x \\
&= \frac{m^2-2m+1}{m} + 2 \ln(|x+1|)\Big]_{\frac{m^2-2m+1}{m^2}}^{0} \\
&= \frac{m^2-2m+1}{m} - 2\ln\left(\frac{2m^2-2m+1}{m^2} \right)
\end{aligned}
$$

6.2 Volumes

Evaluate the integral

$$
\begin{aligned}
\int_1^2 \pi \left(2 - \frac{x}{2}\right)^2 \mathrm{d}x &= \pi \int_1^2 \left(4 - x + \frac{x^2}{4}\right) \mathrm{d}x \\
&= \pi \left[4x - x^2 + \frac{x^3}{12} \right]_1^2 \\
&= \pi \left(4 - 3 + \frac{7}{12} \right) \\
&= \frac{19\pi}{12} \qquad (1)
\end{aligned}
$$

$$\int_1^5 \pi(x-1)\,\mathrm{d}x = \pi \left[\frac{x^2}{2} - x \right]_1^5 = \pi(12-4) = 8\pi \qquad (3)$$

$$\int_0^9 4\pi y \,\mathrm{d}y = 2\pi y^2\Big]_0^9 = 162\pi \qquad (5)$$

$$\int_0^1 \pi \left| x^2 - x^6 \right| \mathrm{d}x = \pi \left[\frac{x^3}{3} - \frac{x^7}{7} \right]_0^1 = \frac{4\pi}{21} \qquad (7)$$

$$\int_{-2}^{2} \pi \left| \frac{x^4}{16} - 25 + 10x^2 - x^4 \right| \mathrm{d}x = 2\pi \int_0^2 \left(\frac{15x^4}{16} - 10x^2 + 25 \right) \mathrm{d}x$$
$$= 2\pi \left[-\frac{10x^3}{3} + 25x + \frac{3x^5}{16} \right]_0^2$$
$$= \frac{88\pi}{3} \tag{8}$$

$$\int_0^1 \pi \left| \left(\sqrt{x} - 1\right)^2 - \left(x^2 - 1\right)^2 \right| \mathrm{d}x = \int_0^1 \pi \left| x - 2\sqrt{x} - x^4 + 2x^2 \right| \mathrm{d}x$$
$$= \int_0^1 \pi \left(x^4 - 2x^2 - x + 2\sqrt{x} \right) \mathrm{d}x$$
$$= \pi \left[\frac{x^5}{5} - \frac{2x^3}{3} - \frac{x^2}{2} + \frac{4\sqrt{x}^3}{3} \right]_0^1$$
$$= \pi \left(\frac{1}{5} - \frac{2}{3} - \frac{1}{2} + \frac{4}{3} \right)$$
$$= \frac{11\pi}{30} \tag{11}$$

$$\int_0^h \left(a + \frac{x}{h}(b-a) \right)^2 \mathrm{d}x = \int_0^h \left(a^2 - \frac{ax}{h}(b-a) + \frac{x^2}{h^2}(b-a)^2 \right) \mathrm{d}x$$
$$= \left[a^2 x - \frac{ax^2}{2h}(b-a) + \frac{x^3}{3h^2}(b-a)^2 \right]_0^h$$
$$= ha^2 - \frac{ha}{2}(b-a) + \frac{h}{3}(b-a)^2$$
$$= ha^2 - \frac{hab}{2} + \frac{ha^2}{2} + \frac{ha^2}{3} - \frac{2hab}{3} + \frac{hb^2}{3}$$
$$= \frac{11ha^2 - 7hab + 2hb^2}{6} \tag{50}$$

$$
\begin{aligned}
A &= \int_{-r}^{r} \left(\pi\left(R+\sqrt{r^2-x^2}\right)^2 - \pi\left(R-\sqrt{r^2-x^2}\right)^2\right) \mathrm{d}x \\
&= \int_{-r}^{r} 4\pi R\sqrt{r^2-x^2}\,\mathrm{d}x \\
&= 2\pi R \int_{-r}^{r} 2\sqrt{r^2-x^2}\,\mathrm{d}x \\
&= 2\pi R \cdot \pi r^2 \\
&= 2\pi^2 R r^2
\end{aligned} \tag{61}
$$

6.4 Work

7. Spring constant: $k = F(4)/4 = 10g/4 = 2.5g$ (lbf/in)
 Work done by stretching the spring from its natural length to 6 in:

$$\int_0^6 kx\,\mathrm{d}x = k\frac{x^2}{2}\bigg]_0^6 = 18k = 45g \text{ (lbf.in)}$$

9. Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm.

$$
\begin{aligned}
\int_0^{0.12} kx\,\mathrm{d}x = 2 &\iff \frac{kx^2}{2}\bigg]_0^{0.12} = 2 \\
&\iff 0.0072k = 2 \\
&\iff k = \frac{2500}{9} \text{ (N/m)}
\end{aligned}
$$

$$
\begin{aligned}
\int_{0.05}^{0.1} kx\,\mathrm{d}x &= \int_{0.05}^{0.1} \frac{2500x}{9}\,\mathrm{d}x \\
&= \frac{1250x^2}{9}\bigg]_{0.05}^{0.1} \\
&= \frac{1250(0.01-0.0025)}{9} \\
&= \frac{25}{24} \text{ (J)}
\end{aligned} \tag{a}
$$

$$x = \frac{F}{k} = \frac{30\cdot 9}{2500} = \frac{27}{250} \text{ (m)} \tag{b}$$

Evaluate the integral

$$\int_0^{50} mgx\,\mathrm{d}x = \left.\frac{25gx^2}{2}\right]_0^{50} = 31250g \text{ (ft.lbf)} \tag{13}$$

$$\begin{aligned}
W &= \lim_{n\to\infty}\sum_{i=1}^{n} F_i \frac{3i}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} m_i g \frac{3i}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} A_i \frac{3}{n}\rho g \frac{3i}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} 3\left(1-\frac{i}{n}\right) 8\left(1-\frac{i}{n}\right) 1000g\frac{3i}{n}\cdot\frac{3}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \frac{80000}{3}\left(3-\frac{3i}{n}\right)^2 \frac{3i}{n}\cdot\frac{3}{n} \\
&= \int_0^3 \frac{80000}{3}\left(9x-6x^2+x^3\right)\,\mathrm{d}x \\
&= \frac{80000}{3}\left[\frac{9x^2}{2}-2x^3+\frac{x^4}{4}\right]_0^3 \\
&= 180000 \text{ (J)}
\end{aligned} \tag{21}$$

$$
\begin{aligned}
W &= \lim_{n\to\infty} \sum_{i=1}^{n} m_i g \frac{8i}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} A_i \frac{8}{n} \rho g \frac{8i}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(6 - 3\frac{i}{n}\right)^2 62.5g \frac{8i}{n} \cdot \frac{8}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1125\pi g}{128} \left(16 - \frac{8i}{n}\right)^2 \frac{8i}{n} \cdot \frac{8}{n} \\
&= \frac{1125\pi g}{128} \int_0^8 (16 - x)^2 \, x \, \mathrm{d}x \\
&= \frac{1125\pi g}{128} \left[128x^2 - \frac{32x^3}{3} + \frac{x^4}{4}\right]_0^8 \\
&= 33000\pi g \text{ (ft.lbf)}
\end{aligned} \tag{23}
$$

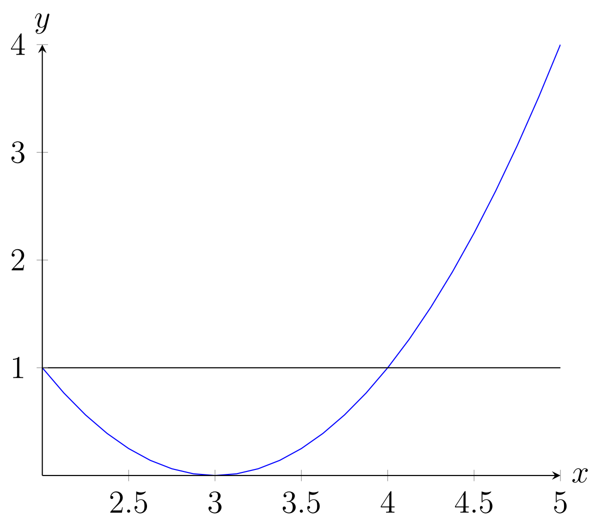
6.5 Average Value of a Function

9. Given the function $f(x) = (x-3)^2$ on $[2, 5]$.

$$f_{ave} = \frac{1}{5-2} \int_2^5 (x-3)^2 \, \mathrm{d}x = \frac{1}{3} \left[\frac{x^3}{3} - 3x^2 + 9x\right]_2^5 = 1 \tag{a}$$

Since $x \in [2, 5]$,

$$f(c) = f_{ave} \iff (c-3)^2 = 1 \iff c = 4 \tag{b}$$



12. Given $f(x) = 2\sin x - \sin 2x$ on $[0, \pi]$.

$$\begin{aligned} f_{ave} &= \frac{1}{\pi}\int_0^\pi (2\sin x - \sin 2x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{\cos 2x}{2} - 2\cos x\right]_0^\pi \\ &= \frac{4}{\pi} \end{aligned}$$

$f(c) = f_{ave} \iff 2\sin x - \sin 2x = \frac{4}{\pi}$, i.e. $x \approx 1.24$ or $x \approx 2.81$ on $[0, \pi]$.

13. Since f is continuous, apply Mean Value Theorem on $[1, 3]$,

$$\exists c \in [1, 3], f(c) = \frac{1}{3-1}\int_1^3 f(x)\,\mathrm{d}x = \frac{8}{2} = 4$$

7 Techniques of Integration

7.1 Integration by Parts

Evaluate the integral

$$\begin{aligned} \int x\cos 5x\,\mathrm{d}x &= \int \frac{x}{5}\,\mathrm{d}\sin 5x \\ &= \frac{x\sin 5x}{5} - \int \sin 5x\,\mathrm{d}\frac{x}{5} \\ &= \frac{x\sin 5x}{5} + \frac{x\cos 5x}{25} + C \end{aligned} \tag{3}$$

$$\begin{aligned} \int e^{2\theta}\sin 3\theta\,\mathrm{d}\theta &= \int \frac{\sin 3\theta}{2}\,\mathrm{d}e^{2\theta} \\ &= \frac{e^{2\theta}\sin 3\theta}{2} - \int \frac{3}{2}e^{2\theta}\cos 3\theta\,\mathrm{d}\theta \\ &= \frac{e^{2\theta}\sin 3\theta}{2} - \frac{3}{2}\int \frac{\cos 3\theta}{2}\,\mathrm{d}e^{2\theta} \\ &= \frac{e^{2\theta}\sin 3\theta}{2} - \frac{3e^{2\theta}\cos 3\theta}{4} - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,\mathrm{d}\theta \\ &= \frac{4}{13}e^{2\theta}\left(\frac{\sin 3\theta}{2} - \frac{\cos 3\theta}{4}\right) + C \\ &= \frac{e^{2\theta}}{13}(2\sin 3\theta - 3\cos 3\theta) + C \end{aligned} \tag{17}$$

9 Differential Equations

9.3 Separable Equations

Solve the equation.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 &\iff y^{-2}\,\mathrm{d}y = x\,\mathrm{d}x \\
&\implies \int \frac{\mathrm{d}y}{y^2} = \int x\,\mathrm{d}x \qquad \text{(for } y \neq 0\text{)} \\
&\iff C_y - \frac{1}{y} = C_x + \frac{x^2}{2} \\
&\iff \frac{C - x^2}{2} = \frac{1}{y} \\
&\iff y = \frac{2}{C - x^2} \qquad (1)
\end{aligned}$$

$$\begin{aligned}
(y + \sin y)\frac{\mathrm{d}y}{\mathrm{d}x} = x + x^3 &\iff \int (y + \sin y)\,\mathrm{d}y = \int (x + x^3)\,\mathrm{d}x \\
&\iff \frac{y^2}{2} - \cos y = \frac{x^2}{2} + \frac{x^4}{4} + C \qquad (5)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y\exp(y + t^2)} &\iff \int ye^y\,\mathrm{d}y = \int te^{-t^2}\,\mathrm{d}t \\
&\iff (y - 1)e^y = C - \frac{1}{2e^{t^2}} \qquad (7)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{2t + \sec^2 t}{2u} &\iff \int 2u\,\mathrm{d}u = \int (2t + \sec^2 t)\,\mathrm{d}t \\
&\iff u^2 = t^2 + \tan t + C \qquad (13)
\end{aligned}$$

Since $u(0) = -5$, $u = -\sqrt{t^2 + \tan t + 25}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = xy &\iff \int \frac{\mathrm{d}y}{y} = \int x\,\mathrm{d}x \qquad \text{(since } y \neq 0\text{)} \\
&\iff \ln|y| = \frac{x^2}{2} + C \\
&\iff |y| = \exp\left(\frac{x^2}{2} + C\right) \qquad (19)
\end{aligned}$$

Since $y(0) = 1$, $y = \exp(x^2/2)$.

$$
\begin{aligned}
y(x) = 2 + \int_2^x [t - ty(t)]\,\mathrm{d}t \implies y - 2 &= \int (x - xy)\,\mathrm{d}x \\
\iff \frac{\mathrm{d}(y-2)}{\mathrm{d}x} &= x - xy \\
\iff \int \frac{\mathrm{d}y}{1-y} &= \int x\,\mathrm{d}x \\
\iff C - \frac{x^2}{2} &= \ln|1-y| \\
\iff y &= 1 \pm \exp\left(C - \frac{x^2}{2}\right) \qquad (33)
\end{aligned}
$$

Since $y(2) = 2$ (which can be trivially obtained from the original condition), $y = 1 + \exp(2 - x^2/2)$.

9.4 Models for Population Growth

3. The Pacific halibut fishery has been modeled by the differential equation

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{M}\right) &\implies \int \left(\frac{1}{y} + \frac{1}{M-y}\right)\mathrm{d}y = \int k\,\mathrm{d}t \\
&\iff \ln|y| - \ln|M-y| = kt + C \\
&\iff \left|\frac{M}{y} - 1\right| = e^{-kt-C} \\
&\iff \frac{M}{y} = 1 \pm e^{-kt-C} \\
&\iff y = \frac{M}{1 \pm e^{-kt-C}} \qquad (*)
\end{aligned}
$$

As $M = 8 \times 10^7$, $k = 0.71$ and $y(0) = 2 \times 10^7$, from $(*)$ we get $\pm e^{-C} = 3$ and thus

$$
y = \frac{M}{1 + 3e^{-kt}}
$$

For $t = 1$, $y \approx 3.2 \times 10^7$. For $y = 4 \times 10^7$, $t = (\ln 3)/0.71$.

5. Suppose a population grows according to a logistic model

$$
\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right) \iff P(t) = \frac{M}{1 \pm e^{-kt-C}}
$$

with initial population $P(0) = 1000$ and carrying capacity $M = 10000$.

Suppose $P(1) = 2500$,

$$\begin{cases} \dfrac{10000}{1 \pm e^{-C}} & = 1000 \\ \dfrac{10000}{1 \pm e^{-k-C}} & = 2500 \end{cases} \iff \begin{cases} \pm e^{-C} & = 9 \\ \pm e^{-k-C} & = 3 \end{cases} \iff \begin{cases} \pm := + \\ C = -\ln 9 \\ k = \ln 3 \end{cases}$$

After another 3 years, the population will be

$$P(3) = \left(t \mapsto \frac{10000}{1 + 3^{2-t}}\right)(1+3) = 9000$$

9.5 Linear Equations

Solve the differential equation.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + y = x &\iff e^x \frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}e^x}{\mathrm{d}x} = xe^x \\ &\iff \int \mathrm{d}ye^x = \int x \,\mathrm{d}e^x \\ &\iff ye^x = e^x(x-1) + C \\ &\iff y = x - 1 + Ce^{-x} \end{aligned} \tag{7}$$

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \sqrt{x} &\iff \int \mathrm{d}xy = \int \sqrt{x}\,\mathrm{d}x \\ &\iff xy = \frac{2x\sqrt{x}}{3} + C \\ &\iff y = \frac{2\sqrt{x}}{3} + \frac{C}{x} \end{aligned} \tag{9}$$

$$\begin{aligned} x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \ln x &\iff \int \mathrm{d}yx^2 = \int \ln x \,\mathrm{d}x \\ &\iff yx^2 = x(\ln x - 1) + C \\ &\iff y = \frac{\ln x - 1}{x} + \frac{C}{x^2} \end{aligned}$$

Since $y(1) = 2$, $C = 3$.

$$\begin{aligned}
L\frac{\mathrm{d}I}{\mathrm{d}t} + RI = \mathcal{E} &\iff e^{Rt/L}\left(\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{R}{L}I\right) = \frac{\mathcal{E}}{L}e^{Rt/L}\\
&\iff \int \mathrm{d}Ie^{Rt/L} = \frac{\mathcal{E}}{L}\int e^{Rt/L}\,\mathrm{d}t\\
&\iff Ie^{Rt/L} = \frac{\mathcal{E}}{R}e^{Rt/L} + C\\
&\iff I = \frac{\mathcal{E}}{R} + \frac{C}{\exp(Rt/L)}
\end{aligned}$$

Since $\mathcal{E} = 40$ V, $L = 2$ H, $R = 10\,\Omega$ and $I(0) = 0$, $I(t) = 4 - 4/\exp 5t$ and $I(0.1) = 4 - 4/\sqrt{e}$.

11 Lazy Evaluation

11.3 The Integral Test and Estimates of Sums

34. Using Leonhard Euler's calculation of the exact sum of the p-series with $p = 2$:

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i^2} = \frac{\pi^2}{6}$$

$$\sum_{n=2}^{\infty}\frac{1}{n^2} = \lim_{n\to\infty}\sum_{i=2}^{n}\frac{1}{i^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i^2} - \frac{1}{1^2} = \frac{\pi^2}{6} - 1 \tag{a}$$

$$\sum_{n=3}^{\infty}\frac{1}{(n+1)^2} = \lim_{n\to\infty}\sum_{i=4}^{n+1}\frac{1}{i^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i^2} - \sum_{i=1}^{3}\frac{1}{i^2} = \frac{\pi^2}{6} - \frac{49}{36} \tag{b}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{4i^2} = \frac{1}{4}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i^2} = \frac{\pi^2}{24} \tag{c}$$

Determine if the series is convergent or divergent using the Integral Test.

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2} \tag{22}$$

$$\begin{aligned}
\int_2^{\infty}\frac{1}{x(\ln x)^2} &= \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^2}\,\mathrm{d}x\\
&= \lim_{t\to\infty}\int_2^t \frac{1}{(\ln x)^2}\,\mathrm{d}\ln x
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{x^2}\,\mathrm{d}x \\
&= \lim_{t\to\infty}\frac{-1}{x}\bigg]_{\ln 2}^{\ln t} \\
&= \lim_{t\to\infty}\left(\frac{1}{\ln 2}-\frac{1}{\ln t}\right) \\
&= \frac{1}{\ln 2}
\end{aligned}
$$

Thus by the Integral Test, the given series is convergent.

$$
\sum_{n=3}^{\infty}\frac{n^2}{e^n} \tag{24}
$$

$$
\begin{aligned}
\int_3^{\infty}\frac{x^2}{e^x} &= \lim_{t\to\infty}\int_3^t\frac{x^2}{e^x}\,\mathrm{d}x \\
&= \lim_{t\to\infty}\int_3^t -x^2\,\mathrm{d}e^{-x} \\
&= \lim_{t\to\infty}\left(\int_3^t e^{-x}\,\mathrm{d}x^2 - x^2e^{-x}\Big]_3^t\right) \\
&= \lim_{t\to\infty}\left(-\int_3^t 2x\,\mathrm{d}e^{-x} + \frac{x^2}{e^x}\bigg]_t^3\right) \\
&= \lim_{t\to\infty}\left(\int_3^t e^{-x}\,\mathrm{d}2x + \left[\frac{2x}{e^x}+\frac{x^2}{e^x}\right]_t^3\right) \\
&= \lim_{t\to\infty}\left(2\int_3^t e^{-x}\,\mathrm{d}x + \frac{2x+x^2}{e^x}\bigg]_t^3\right) \\
&= \lim_{t\to\infty}\frac{2+2x+x^2}{e^x}\bigg]_t^3 \\
&= \lim_{t\to\infty}\left(\frac{17}{e^3}-\frac{2+2t+t^2}{e^t}\right) \\
&= \frac{17}{e^3}
\end{aligned}
$$

Thus by the Integral Test, the given series is convergent.

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}} \tag{27}$$

Since $x \mapsto \cos(\pi x)/\sqrt{n}$ is neither positive (e.g. $\cos 3\pi/\sqrt{3} = -1$) nor ultimately decreasing, the Integral Test cannot be used to determine whether the series is convergent.

11.4 The Comparison Test

Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n^2} \tag{25}$$

We use the Limit Comparison Test with

$$a_n = \frac{\sqrt{n^4+1}}{n^3+n^2} \qquad b_n = \frac{1}{n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^2+n} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^4}}}{1+\frac{1}{n}} = 1 > 0$$

Since this limit exists and $\sum \frac{1}{n}$ is divergent (p-series with $p = 1$), the given series diverges by the Limit Comparison Test.

$$\sum_{n=1}^{\infty} \frac{1}{n!} \tag{29}$$

$$\lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

Thus by the Ratio Test, the given series is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \tag{30}$$

$$\frac{n!}{n^n} = \frac{2}{n^2} \cdot \frac{n!}{2n^{n-2}} \leq \frac{2}{n^2}$$

Since both $\sum n!/n^n$ and $\sum 2/n^2$ are series with positive terms and $\sum 2/n^2$ converges because it is a constant time of p-series with $p = 2$, by the Comparison Test, $\sum n!/n^n$ is convergent.

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}} \tag{32}$$

We use the Limit Comparison Test with

$$a_n = \frac{1}{n\sqrt[n]{n}} \qquad b_n = \frac{1}{n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}}$$

Since $\frac{1}{\sqrt[n]{n}}' = \frac{1-\ln n}{n^2\sqrt[n]{n}}$ is negative on (e,∞), $n \mapsto \frac{1}{\sqrt[n]{n}}$ is ultimately decreasing. Additionally, $\frac{1}{\sqrt[n]{n}} \geq \frac{1}{\sqrt[n]{1}} = 1$ on this interval, thus

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \inf\left\{\frac{1}{\sqrt[n]{n}} : n \in \mathbb{N}_3\right\} = 1 > 0$$

Therefore, the given series diverges by the Limit Comparison Test, as $\sum \frac{1}{n}$ is divergent (p-series with $p = 1$).

11.5 Alternating Series

Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!} \tag{19}$$

Since $\frac{(n+1)^{n+1}}{(n+1)!} > \frac{n^n}{n!}$, the given alternating series diverges.

$$\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right) \tag{20}$$

For all n,

$$\begin{aligned} n^2 + 2n < n^2 + 2n + 1 &\iff \sqrt{n(n+2)} < n+1 \\ &\iff n + \sqrt{n(n+2)} + n + 2 < 4n + 4 \\ &\iff \sqrt{n+2} + \sqrt{n} < 2\sqrt{n+1} \\ &\iff \sqrt{n+2} - \sqrt{n+1} < \sqrt{n+1} - \sqrt{n} \end{aligned} \tag{i}$$

$$\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{\sqrt{1 + 1/\sqrt{n}} + 1} = 0 \tag{ii}$$

Thus, by the Alternating Series Test, the given series is convergent.

11.6 Absolute Convergence

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1} \right)^{5n} \tag{22}$$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1} \right)^{5n}} = \lim_{n\to\infty} \left(\frac{2n}{n+1} \right)^{5} = \left(\lim_{n\to\infty} \frac{2}{1+1/n} \right)^{5} = 32 > 1$$

Thus the given series diverges by the Root Test.

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{2i}{3i+2} \tag{30}$$

$$\lim_{n\to\infty} \frac{\prod_{i=1}^{n+1} \frac{2i}{3i+2}}{\prod_{i=1}^{n} \frac{2i}{3i+2}} = \lim_{n\to\infty} \frac{2n+2}{3n+5} = \lim_{n\to\infty} \frac{2+2/n}{3+5/n} = \frac{2}{3} < 1$$

Thus by the Ratio Test, the given series is absolutely convergent.

11.8 Power Series

Find the radius of convergence and the interval of convergence of the series.

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{14}$$

Let $a_n = (-1)^n x^{2n+1}/(2n+1)!$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^2}{4n^2+10n+6} \right| = 0 < 1$$

Thus by the Ratio Test, the series is convergent for all x and the radius of convergence is $R = \infty$.

$$\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}} \tag{17}$$

Let $a_n = 3^n (x+4)^n / \sqrt{n}$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3(x+4)\sqrt{n}}{\sqrt{n+1}} \right| = 3|x+4|$$

Using the Ratio Test, we see that the series converges if $|x+4| < 1/3$ and it diverges if $|x+4| > 1/3$, thus the radius of convergence is $R = 1/3$.

When $|x+4| = 1/3$, the series is either $\sum(-3)^n/\sqrt{n}$ or $\sum 3^n/\sqrt{n}$, both of which diverge by the Test for Divergence. Therefore the interval of convergence is $(-13/3, -11/3)$.

$$\sum_{n=1}^{\infty} \frac{b^n}{\ln n}(x-a)^n, \qquad b > 0 \tag{22}$$

Let $a_n = b^n(x-a)^n/\ln n$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{b(x-a)\ln n}{\ln(n+1)}\right| = b|x-a|$$

Using the Ratio Test, we see that the series converges if $|x-a| < b^{-1}$ and it diverges if $|x-a| > b^{-1}$, thus the radius of convergence is $R = b^{-1}$.

When $|x-a| = b^{-1}$, the series is $\sum(\pm b)^n/\ln n$, which diverges by the Test for Divergence. Therefore the interval of convergence is $(a-b^{-1}, a+b^{-1})$.

$$\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2} \tag{26}$$

Let $a_n = x^{2n}/n/(\ln n)^2$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^2 n \ln^2 n}{(n+1)\ln^2(n+1)}\right| = x^2$$

Using the Ratio Test, we see that the series converges if $|x| < 1$ and it diverges if $|x| > 1$, therefore the radius of convergence is $R = 1$.

When $x = \pm 1$, $a_n = n^{-1}/(\ln n)^2$, which is defined by a continuous, positive and decreasing function $x \mapsto x^{-1}/(\ln x)^2$ on $[2, \infty)$.

$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\ln x)^2}\,\mathrm{d}x &= \lim_{t\to\infty}\int_2^t \frac{1}{(\ln x)^2}\,\mathrm{d}\ln x \\ &= \lim_{t\to\infty}\int_{\ln 2}^{\ln t} \frac{1}{x^2}\,\mathrm{d}x \\ &= \lim_{t\to\infty} \frac{1}{3x^3}\bigg]_{\ln t}^{\ln 2} \\ &= \frac{1}{3(\ln 2)^3}\end{aligned}$$

By the Integral Test, $\sum_{n=2}^{\infty} n^{-1}/(\ln n)^2$ converges, and thus the interval if of convergence of the given power series is $[-1, 1]$.

11.9 Representations of Functions as Power Series

Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{5}{1-4x^2} = 5\sum_{n=0}^{\infty}\left(4x^2\right)^n = \sum_{n=0}^{\infty} 5\cdot 2^{2n}x^{2n} \tag{4}$$

Interval of convergence is $(-1, 1)$.

$$f(x) = \frac{x^2}{a^3 - x^3} = \frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\frac{x}{a}\right)^{3n} = \sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}} \tag{10}$$

Interval of convergence is $(-a, a)$.

$$\begin{aligned} f(x) &= \frac{x+2}{2x^2 - x - 1} \\ &= \frac{1}{x-1} - \frac{1}{2x+1} \\ &= -\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}(-2x)^n \\ &= \sum_{n=0}^{\infty}(-1-(-2)^n)x^n \end{aligned} \tag{12}$$

Interval of convergence is $(-1, 1) \cap (-1/2, 1/2) = (-1/2, 1/2)$.

40. Find the sum of the series when $|x| < 1$.

$$\begin{aligned} \sum_{n=1}^{\infty} nx^{n-1} &= \sum_{n=1}^{\infty} x^{n-1} + \sum_{n=1}^{\infty}(n-1)x^{n-1} \\ &= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} nx^n \\ &= \frac{1}{1-x} + x\sum_{n=1}^{\infty} nx^{n-1} \\ &= \frac{1}{(1-x)^2} \end{aligned} \tag{a}$$

$$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2} \tag{b.i}$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \left(x \mapsto \frac{x}{(1-x)^2}\right)\left(\frac{1}{2}\right) = 2 \tag{b.ii}$$

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)x^n &= \sum_{n=2}^{\infty} 2(n-1)x^n + \sum_{n=2}^{\infty}(n-1)(n-2)x^n \\
&= 2\sum_{n=1}^{\infty}(n-1)x^n + x\sum_{n=1}^{\infty} n(n-1)x^n \\
&= 2\left(\sum_{n=1}^{\infty} nx^n + 1 - \sum_{n=0}^{\infty} x^n\right) : (1-x) \\
&= 2\left(\frac{x}{(1-x)^2} + 1 - \frac{1}{1-x}\right) : (1-x) \\
&= \frac{2x^2}{(1-x)^3}
\end{aligned} \tag{c.i}$$

$$\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = \left(x \mapsto \frac{2x^2}{(1-x)^3}\right)\left(\frac{1}{2}\right) = 4 \tag{c.ii}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=2}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} nx^n = 4 + 2 = 6 \tag{c.iii}$$

11.10 Taylor and Maclaurin Series

Find the Taylor series for f centered at the given value of a and the associative radius of convergence.

$$f(x) = \ln x, \qquad a = 2 \tag{15}$$

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= \ln 2 + \sum_{n=1}^{\infty} \left(x \mapsto \binom{-1}{n-1}\frac{1}{nx^n}\right)(2) \cdot (x-2)^n \\
&= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(x-2)^n}{n2^n}
\end{aligned}$$

Let $a_n = (-1)^{n-1}(x-2)^n/(n2^n)$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{2n+2}|x-2| = \frac{|x-2|}{2}$$

Using the Ratio Test, we see $f(x) = \ln 2 + \sum a_n$ converges if $|x-2| < 2$ and it diverges if $|x-2| > 2$, therefore the associative radius of convergence is $R = 2$.

$$f(x) = \frac{1}{x}, \qquad a = -3 \tag{16}$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{n=0}^{\infty}\left(x \mapsto \binom{-1}{n}\frac{1}{x^{n+1}}\right)(-3)\cdot(x+3)^n \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(x+3)^n}{(-3)^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{-(x+3)^n}{3^{n+1}} \end{aligned}$$

Let $a_n = (x+3)^n/3^{n+1}$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x+3|}{3} = \frac{|x+3|}{3}$$

Using the Ratio Test, we see that the series converges if $|x+3| < 3$ and it diverges if $|x+3| > 3$, therefore the associative radius of convergence is $R = 3$.

$$f(x) = \sin x, \qquad a = \frac{\pi}{2} \tag{18}$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{n=0}^{\infty}\frac{\cos\frac{n\pi}{2}}{n!}\left(x-\frac{\pi}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n} \end{aligned}$$

Let $a_n = (-1)^n (x - \pi/2)^{2n}/(2n)!$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(x-\pi/2)^2}{(2n+2)(2n+1)} = 0 < 1$$

Using the Ratio Test, we see that the series converges for all x, thus the associative radius of convergence is $R = \infty$.

$$f(x) = \sqrt{x}, \qquad x = 16 \tag{20}$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{n=0}^{\infty} \left(x \mapsto \binom{\frac{1}{2}}{n} x^{1/2-n} \right)(16) \cdot (x-16)^n \\ &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{4(x-16)^n}{16^n} \end{aligned}$$

Let $a_n = 4\binom{1/2}{n}(x-16)^n/16^n$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{1/2 - n}{n+1} \right| \frac{|x-16|}{16} = \frac{|x-16|}{16}$$

Using the Ratio Test, we see that the series converges if $|x - 16| < 16$ and it diverges if $|x - 16| > 16$, therefore the associative radius of convergence is $R = 16$.

55. Use series to evaluate the limit.

$$\begin{aligned} \lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} &= \lim_{x\to 0} \frac{x - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}}{x^2} \\ &= \lim_{x\to 0} \frac{\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{n}}{x^2} \\ &= \lim_{x\to 0} \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n+2} \\ &= \lim_{x\to 0} \frac{1}{2} + \lim_{x\to 0} \sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n+2} \\ &= \frac{1}{2} \end{aligned}$$

Find the sum of the series.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-\ln 2)^n}{n!} &= \left(x \mapsto \sum_{n=0}^{\infty}\frac{x^n}{n!}\right)(-\ln 2)\\ &= (x \mapsto e^x)\left(\ln\frac{1}{2}\right)\\ &= \exp\left(\ln\frac{1}{2}\right)\\ &= \frac{1}{2} \qquad (68)\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)2^{2n+1}} &= \left(x \mapsto \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}}{2n+1}\right)\left(\frac{1}{2}\right)\\ &= \left(x \mapsto \tan^{-1} x\right)\left(\frac{1}{2}\right)\\ &= \tan^{-1}\frac{1}{2} \qquad (70)\end{aligned}$$

72. If $f(x) = (1+x^3)^{30}$, what is $f^{(58)}(0)$?

$$\begin{aligned}f(x) = \sum_{n=0}^{30}\binom{30}{n}x^{3n} &\implies f'(x) = \sum_{n=0}^{30}\binom{30}{n}x^{3n-1}3n\\ &\implies f''(x) = \sum_{n=1}^{30}\binom{30}{n}x^{3n-2}3n(3n-1)\\ &\implies f^{(58)}(x) = \sum_{n=20}^{30}\binom{30}{n}x^{3n-58}\prod_{i=0}^{57}(3n-i)\\ &\implies f^{(58)}(0) = \sum_{n=20}^{30}\binom{30}{n}0^{3n-58}\prod_{i=0}^{57}(3n-i) = 0\end{aligned}$$