Cuculutu Homework

Nguyễn Gia Phong

Summer 2019

12 Vectors and the Geometry of Space

12.1 Three-Dimenstional Coordinate Systems

37. The region consisting of all points between the spheres of radius r and R centered at origin:

$$r^2 < x^2 + y^2 + z^2 < R^2 \qquad (r < R)$$

12.2 Vectors

38. The gravitational force enacting the chain whose tension T at each end has magnitude 25 N and angle 37° to the horizontal is

$$\mathbf{P} = 2\text{proj}_{\hat{\mathbf{P}}}\mathbf{T} = 2T\sin 37°\hat{\mathbf{P}} \approx 30\hat{\mathbf{P}}$$

Therefore the weight of the chain is approximately 30 N.

47. Given $\mathbf{r_0} = \langle x_0, y_0, z_0\rangle$.

Let $\mathbf{r} = \langle x, y, z\rangle$,

$$|\mathbf{r} - \mathbf{r_0}| = 1 \iff (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1$$

Thus the set of all points (x, y, z) is an unit sphere whose center is (x_0, y_0, z_0).

12.3 The Dot Product

25. Given a triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, $R(6, -2, -5)$.

Since $\overrightarrow{PQ} \cdot \overrightarrow{QR} = 1 \cdot 4 + 3(-2) + (-2)(-1) = 0$, PQR is a right triangle.

26. Given $\mathbf{u} = \langle 2, 1, -1 \rangle$ and $\mathbf{v} = \langle 1, x, 0 \rangle$.

$$\begin{aligned}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\cdot|\mathbf{v}|} = \cos 45^\circ &\iff \frac{2+x}{\sqrt{6\left(x^2+1\right)}} = \frac{1}{\sqrt{2}} \\ &\iff 2x^2+8x+8 = 6x^2+6 \\ &\iff 4x^2-8x-2=0 \\ &\iff x = 1 \pm \sqrt{\frac{3}{2}}\end{aligned}$$

27. Find a unit vector that is orthogonal to both $\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\hat{\mathbf{i}}+\hat{\mathbf{k}}$.

A vector that is orthogonal to both of these vectors:

$$(\hat{\mathbf{i}}+\hat{\mathbf{j}})\times(\hat{\mathbf{i}}+\hat{\mathbf{k}}) = \hat{\mathbf{i}}\times\hat{\mathbf{i}}+\hat{\mathbf{i}}\times\hat{\mathbf{k}}+\hat{\mathbf{j}}\times\hat{\mathbf{i}}+\hat{\mathbf{j}}\times\hat{\mathbf{k}} = 0-\hat{\mathbf{j}}-\hat{\mathbf{k}}+\hat{\mathbf{i}} = \hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}$$

Normalize the result we get the unit vector $\frac{1}{\sqrt{3}}\left(\hat{\mathbf{i}}-\hat{\mathbf{j}}-\hat{\mathbf{k}}\right)$ which is orthogonal to both $\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\hat{\mathbf{i}}+\hat{\mathbf{k}}$.

28. Find two unit vectors that make an angle of 60° with $\mathbf{v} = \langle 3, 4 \rangle$.

Let $\mathbf{u} = \langle x, y \rangle$ be an unit vector, $|\mathbf{u}| = \sqrt{x^2+y^2} = 1$. $\mathbf{u}$ makes with $\mathbf{v}$ an angle of 60° if and only if

$$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\cdot|\mathbf{v}|} = \cos 60^\circ \iff \frac{3x+4y}{\sqrt{3^2+4^2}} = \frac{1}{2} \iff 6x+8y=5$$

Since $x^2+y^2=1$, $\mathbf{u} = \left\langle 0.3 \pm 0.4\sqrt{3}, 0.4 \mp 0.3\sqrt{3} \right\rangle$.

53. Given a point $P_1\left(x_1, y_1\right)$ and a line $d: ax+by+c=0$.

Let $P\left(x_0, y_0\right)$ be any point satisfying $ax_0+by_0+c=0$, distance (d, P_1) is component of $\mathbf{u} = \overrightarrow{PP_1} = \langle x_1-x_0, y_1-y_0 \rangle$ along the normal of the line $\mathbf{n} = \langle a, b \rangle$:

$$\begin{aligned}\text{comp}_{\mathbf{u}}\mathbf{n} = \frac{|\mathbf{n}\cdot\mathbf{u}|}{|\mathbf{n}|} &= \frac{\left|a\left(x_1-x_0\right)+b\left(y_1-y_0\right)\right|}{\sqrt{a^2+b^2}} = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}} \\ \Longrightarrow \text{distance}\,(3x-4y+5=0, (-2,3)) &= \frac{|3(-2)+(-4)3+5|}{\sqrt{3^2+(-4)^2}} = \frac{13}{5}\end{aligned}$$

12.4 The Cross Product

18. Given $\mathbf{a} = \langle 1, 0, 1 \rangle$, $\mathbf{b} = \langle 2, 1, -1 \rangle$ and $\mathbf{c} = \langle 0, 1, 3 \rangle$.

$$\begin{cases}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \langle 1,0,1\rangle \times \langle 4,-6,2\rangle = \langle 6,2,-6\rangle \\ (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \langle -1,3,1\rangle \times \langle 0,1,3\rangle = \langle 8,3,-1\rangle\end{cases}$$

$$\Longrightarrow \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$$

38. Given $A(1,3,2)$, $B(3,-1,6)$, $C(5,2,0)$ and $D(3,6,-4)$.

$$\begin{aligned}\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right) &= \langle 2,-4,4\rangle\cdot(\langle 4,-1,-2\rangle\times\langle 2,3,-6\rangle)\\ &= \langle 2,-4,4\rangle\cdot\langle 12,20,14\rangle\\ &= 24-80+56\\ &= 0\end{aligned}$$

Thus $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, which means A, B, C and D are coplanar.

39. The magnitude of the torque about P:

$$\begin{aligned}|\boldsymbol{\tau}| &= |\mathbf{r}\times\mathbf{F}|\\ &= |-\mathbf{r}\times -\mathbf{F}|\\ &= |\mathbf{r}|\cdot|\mathbf{F}|\cdot\sin(70^\circ+10^\circ)\\ &= 0.18\cdot 60\cdot\sin 80^\circ\\ &\approx 10.6 \qquad (\mathrm{N}\cdot\mathrm{m})\end{aligned}$$

14 Partial Derivatives

14.2 Limits et Continuity

Determine the set of points at which the function is continuous.

$$F(x,y)=\frac{1+x^2+y^2}{1-x^2-y^2} \tag{31}$$

F is a rational function, hence it is continuous on its domain

$$D_F=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, x^2+y^2\neq 1\right\}$$

$$H(x,y)=\frac{e^x+e^y}{e^{xy}-1} \tag{32}$$

Since H is a ratio of sums of exponential functions, it is continuous on its domain

$$D_H=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, xy\neq 0\right\}$$

$$f(x,y)=\begin{cases}\frac{x^2y^3}{2x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 1 & \text{if } (x,y)=(0,0)\end{cases} \tag{37}$$

On $\mathbb{R}^2\backslash(0,0)$, because $2x^2+y^2 \geq 3x^2|y|$ (AM-GM inequality)

$$0 \leq \left|\frac{x^2y^3}{2x^2+y^2}\right| \leq \left|\frac{x^2y^3}{3x^2|y|}\right| = \frac{y^2}{3}$$

Since $0 \to 0$ and $y^2 \to 0$ as $(x,y) \to (0,0)$, by applying the Squeeze Theorem, $|f(x,y)| \to 0$ as $(x,y) \to (0,0)$.

It is trivial on $\mathbb{R}^2\backslash(0,0)$ that $-|f(x,y)| \leq f(x,y) \leq |f(x,y)|$. Thus by again applying the Squeeze Theorem, we get

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 \neq 1 = f(0,0)$$

Therefore, the rational function f is only continuous on $\mathbb{R}^2\backslash(0,0)$.

14.3 Partial Derivatives

29. Find the first partial derivatives of the function

$$\begin{aligned}
F(x,y) &= \int_y^x \cos\left(e^t\right)\, \mathrm{d}t \\
&= \int_y^x \frac{1}{e^t}\, \mathrm{d}\sin\left(e^t\right) \\
&= \int_{e^y}^{e^x} \frac{1}{t}\, \mathrm{d}\sin t \\
&= \int_{e^y}^{e^x} \frac{\cos t}{t}\, \mathrm{d}t \\
&= \sum_{n=0}^{\infty} \int_{e^y}^{e^x} (-1)^n \frac{t^{2n-1}}{(2n)!}\, \mathrm{d}t \\
&= \left[\ln t + \sum_{n=1}^{\infty} \frac{(-t)^{2n}}{2n(2n)!}\right]_{e^y}^{e^x} \\
&= x - y + \sum_{n=1}^{\infty} \frac{(-e^x)^{2n} - (-e^y)^{2n}}{2n(2n)!}
\end{aligned}$$

$$\frac{\partial F}{\partial x} = -1 + \sum_{n=1}^{\infty} \frac{2n\left(-e^x\right)^{2n}}{2n(2n)!} = \sum_{n=0}^{\infty} \frac{\left(-e^x\right)^{2n}}{(2n)!} = \cos\left(-e^x\right) = \cos\left(e^x\right)$$

$$\frac{\partial F}{\partial y} = 1 + \sum_{n=1}^{\infty} \frac{-2n\left(-e^y\right)^{2n}}{2n(2n)!} = -\sum_{n=0}^{\infty} \frac{\left(-e^x\right)^{2n}}{(2n)!} = -\cos\left(-e^x\right) = -\cos\left(e^x\right)$$

48. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

$$x^2 - y^2 + z^2 - 2z = 4$$

$$\begin{cases} 2x + 2z\dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial x} = 0 \\ -2y + 2z\dfrac{\partial z}{\partial y} - 2\dfrac{\partial z}{\partial y} = 0 \end{cases} \implies \begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{x}{1-z} \\ \dfrac{\partial z}{\partial y} = \dfrac{y}{z-1} \end{cases}$$

65&67. Find the indicated partial derivative.

$$\begin{aligned} \frac{\partial^3}{\partial z \partial y \partial x} e^{xyz^2} &= \frac{\partial^2}{\partial z \partial y} yz^2 e^{xyz^2} \\ &= \frac{\partial}{\partial z} xyz^4 e^{xyz^2} \\ &= 2x^2y^2z^5 e^{xyz^2} \end{aligned} \tag{65}$$

$$\begin{aligned} \frac{\partial^3}{\partial r^2 \partial \theta} e^{r\theta} \sin\theta &= \frac{\partial^2}{\partial r^2} \left(re^{r\theta} \sin\theta + e^{r\theta} \cos\theta \right) \\ &= \frac{\partial}{\partial r} \left(r\theta e^{r\theta} \sin\theta + \theta e^{r\theta} \cos\theta \right) \\ &= \theta^2 e^{r\theta} (r \sin\theta + \cos\theta) \end{aligned} \tag{67}$$

53. Find all the second partial derivatives of the function $f(x, y) = x^3y^5 + 2x^4y$.

First partial derivatives of f:

$$\begin{aligned} f_x &= 3x^2y^5 + 8x^3y \\ f_y &= 5x^3y^4 + 2x^4 \end{aligned}$$

Second partial derivatives:

$$\begin{aligned} f_{xx} &= 6xy^5 + 24x^2y \\ f_{xy} &= f_{yx} = 15x^2y^4 + 8x^3 \\ f_{yy} &= 20x^3y^3 \end{aligned}$$

80. Given $u = \exp\left(\sum_{i=1}^n a_i x_i\right)$, where $\sum_{i=1}^n a_i^2 = 1$.

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^n \frac{\partial a_i u}{\partial x_i} = \sum_{i=1}^n a_i^2 u = u$$

14.4 Tangent Planes

Find an equation of the tangent plane to the given suface at the specified point.

$$z = 3y^2 - 2x^2 + x, \qquad (2, -1, -3) \tag{1}$$

$$\begin{aligned}
&z + 3 = \frac{\partial z}{\partial x}(2, -1)(x - 2) + \frac{\partial z}{\partial y}(2, -1)(y + 1) \\
\iff &z + 3 = ((x, y) \mapsto 1 - 4x)(2, -1)(x - 2) + ((x, y) \mapsto 6y)(2, -1)(y + 1) \\
\iff &z + 3 = 17 - 8x - 6y - 6 \\
\iff &8x + 6y + z = 8
\end{aligned}$$

$$z = 3(x - 1)^2 + 2(y + 3)^2 + 7, \qquad (2, -2, 12) \tag{2}$$

$$\begin{aligned}
&z - 12 = \frac{\partial z}{\partial x}(2, -2)(x - 2) + \frac{\partial z}{\partial y}(2, -2)(y + 2) \\
\iff &z - 12 = ((x, y) \mapsto 6x - 6)(2, -2)(x - 2) + ((x, y) \mapsto 4y + 12)(2, -2)(y + 2) \\
\iff &z - 12 = 6x - 12 + 4y + 8 \\
\iff &6x + 4y - z + 8 = 0
\end{aligned}$$

$$z = \sqrt{xy}, \qquad (1, 1, 1) \tag{3}$$

$$\begin{aligned}
&z - 1 = \frac{\partial z}{\partial x}(1, 1)(x - 1) + \frac{\partial z}{\partial y}(1, 1)(y - 1) \\
\iff &z - 1 = \left((x, y) \mapsto \sqrt{\frac{y}{4x}}\right)(1, 1)(x - 1) + \left((x, y) \mapsto \sqrt{\frac{x}{4y}}\right)(1, 1)(y - 1) \\
\iff &2z - 2 = x - 1 + y - 1 \\
\iff &x + y - 2z = 0
\end{aligned}$$

14.5 The Chain Rule

4. Use the Chain Rule to find $\mathrm{d}z/\mathrm{d}t$.

$$z = \arctan \frac{y}{x}, \qquad x = e^t, \qquad y = 1 - e^{-t}$$

$$
\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}\\
&= \frac{\partial \arctan(y/x)}{\partial x}\cdot\frac{\mathrm{d}e^t}{\mathrm{d}t}+\frac{\partial \arctan(y/x)}{\partial y}\cdot\frac{\mathrm{d}\,(1-e^{-t})}{\mathrm{d}t}\\
&= \frac{x^2}{y^2+x^2}\left(\frac{\partial(y/x)}{\partial x}e^t+\frac{\partial(y/x)}{\partial y}e^{-t}\right)\\
&= \frac{x^2}{y^2+x^2}\left(\frac{-y}{x^2}e^t+\frac{1}{x}e^{-t}\right)\\
&= \frac{xe^{-t}-ye^t}{y^2+x^2}\\
&= \frac{1-e^t+1}{e^{2t}+e^{-2t}-2e^{-t}+1}\\
&= \frac{e^{2t}-e^{3t}}{e^{4t}+e^{2t}-2e^t+1}
\end{aligned}
$$

9&11. Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

$$z=\sin\theta\cos\phi,\qquad \theta=st^2,\qquad \phi=s^2t \tag{9}$$

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}+\frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial s}=t^2\cos\theta\cos\phi-2st\sin\theta\sin\phi\\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}+\frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial t}=2st\cos\theta\cos\phi-t^2\sin\theta\sin\phi
\end{aligned}
$$

$$e^r\cos\theta,\qquad r=st,\qquad \theta=\sqrt{s^2+t^2} \tag{11}$$

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial r}\frac{\partial r}{\partial s}+\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}=e^r t\cos\theta-e^r\sin\theta\frac{s}{\sqrt{s^2+t^2}}=e^{st}\left(t\cos\theta-\frac{s\sin\theta}{\sqrt{s^2+t^2}}\right)\\
\frac{\partial z}{\partial t} &= e^{st}\left(s\cos\theta-\frac{t\sin\theta}{\sqrt{s^2+t^2}}\right)
\end{aligned}
$$

13. Suppose f is a differentiable function of $g(t)$ and $h(t)$, satisfying

$$\begin{aligned} g(3) &= 2 \\ \frac{\mathrm{d}g}{\mathrm{d}t}(3) &= 5 \\ \frac{\partial f}{\partial g}(2,7) &= 6 \\ h(3) &= 7 \\ \frac{\mathrm{d}h}{\mathrm{d}t}(3) &= -4 \\ \frac{\partial f}{\partial h}(2,7) &= -8 \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t}(3) &= \frac{\partial f}{\partial g}(g(3),h(3)) \cdot \frac{\mathrm{d}g}{\mathrm{d}t}(3) + \frac{\partial f}{\partial h}(g(3),h(3)) \cdot \frac{\mathrm{d}h}{\mathrm{d}t}(3) \\ &= \frac{\partial f}{\partial g}(2,7) \cdot 5 + \frac{\partial f}{\partial h}(2,7) \cdot (-4) \\ &= 6 \cdot 5 + (-8)(-4) \\ &= 62 \end{aligned}$$

14. Let $W(s,t) = F(u(s,t), v(s,t))$, where F, u and v are differentiable, and

$$\begin{aligned} u(1,0) &= 2 \\ u_s(1,0) &= -2 \\ u_t(1,0) &= 6 \\ F_u(2,3) &= -1 \\ v(1,0) &= 3 \\ v_s(1,0) &= 5 \\ v_t(1,0) &= 4 \\ F_v(2,3) &= 10 \end{aligned}$$

$$\begin{aligned} W_s(1,0) &= F_u(u(1,0),v(1,0))u_s(1,0) + F_v(u(1,0),v(1,0))v_s(1,0) \\ &= F_u(2,3)(-2) + F_v(2,3) \cdot 5 \\ &= (-1)(-2) + 10 \cdot 5 \\ &= 22 \\ W_t(1,0) &= F_u(u(1,0),v(1,0))u_t(1,0) + F_v(u(1,0),v(1,0))v_t(1,0) \\ &= F_u(2,3) \cdot 6 + F_v(2,3) \cdot 4 \\ &= -1 \cdot 6 + 10 \cdot 4 \\ &= 34 \end{aligned}$$

17. Assume all functions are differentiable, write out the Chain Rule.

$$u = f(x(r,s,t), y(r,s,t))$$

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} \\ \dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} \\ \dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t} \end{cases}$$

23. Use the Chain Rule to find $\partial w/\partial r$ and $\partial w/\partial \theta$ when $r = 2$ and $\theta = \pi/2$, given

$$w = xy + yz + zx, \qquad x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = r\theta$$

$$\begin{cases} \dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial r} \\ \dfrac{\partial w}{\partial \theta} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial \theta} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial \theta} \end{cases}$$

$$\iff \begin{cases} \dfrac{\partial w}{\partial r} = (y+z)\cos\theta + (x+z)\sin\theta + (y+x)\theta \\ \dfrac{\partial w}{\partial \theta} = -(y+z)r\sin\theta + (x+z)r\cos\theta + (y+x)r \end{cases}$$

For $(r,\theta) = (2, \pi/2)$

$$\begin{cases} \dfrac{\partial w}{\partial r} = x + z + (y+x)\dfrac{\pi}{2} \\ \dfrac{\partial w}{\partial \theta} = 2x - 2z \end{cases}$$

$$\iff \begin{cases} \dfrac{\partial w}{\partial r} = 2\cos\dfrac{\pi}{2} + 2\dfrac{\pi}{2} + 2\left(\sin\dfrac{\pi}{2} + \cos\dfrac{\pi}{2}\right)\dfrac{\pi}{2} \\ \dfrac{\partial w}{\partial \theta} = 4\cos\dfrac{\pi}{2} - 4\dfrac{\pi}{2} \end{cases}$$

$$\iff \frac{\partial w}{\partial r} = -\frac{\partial w}{\partial \theta} = 2\pi$$

27. Find $\mathrm{d}y/\mathrm{d}x$.

$$y\cos x = x^2 + y^2 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial}{\partial x}(x^2 + y^2 - y\cos x)}{\frac{\partial}{\partial y}(x^2 + y^2 - y\cos x)} = \frac{y\sin x + 2x}{\cos x - 2y}$$

31. Find $\partial z/\partial x$ and $\partial z/\partial y$.

$$x^2+2y^2+3z^2=1 \Longrightarrow \begin{cases} \dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial}{\partial x}\left(x^2+2y^2+3z^2-1\right)}{\frac{\partial}{\partial z}\left(x^2+2y^2+3z^2-1\right)} = -\dfrac{x}{3z} \\ \dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial}{\partial y}\left(x^2+2y^2+3z^2-1\right)}{\frac{\partial}{\partial z}\left(x^2+2y^2+3z^2-1\right)} = -\dfrac{2y}{3z} \end{cases}$$

36. Wheat production W in a given year depends on the average temperature T and the annual rainfall R. At current production levels, $\partial W/\partial T = -2$ and $\partial W/\partial R = 8$. Estimate the current rate of change of wheat production, given $\mathrm{d}T/\mathrm{d}t = 0.15$ and $\mathrm{d}R/\mathrm{d}t = -0.1$.

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\partial W}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\partial W}{\partial R}\frac{\mathrm{d}R}{\mathrm{d}t} = (-1)0.15 + 8(-0.1) = -0.95$$

40. Use Ohm's Law, $V = IR$, to find how the current I is changing at the moment when $R = 400\,\Omega$, $I = 0.08$ A, $\mathrm{d}V/\mathrm{d}t = 0.01$ V/s, and $\mathrm{d}R/\mathrm{d}t = 0.03\,\Omega$/s.

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\partial(V/R)}{\partial V}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\partial(V/R)}{\partial R}\frac{\mathrm{d}R}{\mathrm{d}t} \\ &= \frac{1}{R}(-0.01) - \frac{V}{R^2}0.03 \\ &= \frac{-0.01}{400} - \frac{0.03I}{R} \\ &= \frac{-1}{40000} - \frac{0.03\cdot 0.08}{400} \\ &= \frac{-31}{1000000}\,(\mathrm{A/t}) \\ &= -31\,(\mu\mathrm{A/t}) \end{aligned}$$

42. The rate of change of production:

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &= \frac{\partial\left(1.47L^{0.65}K^{0.35}\right)}{\partial L}\frac{\mathrm{d}L}{\mathrm{d}t} + \frac{\partial\left(1.47L^{0.65}K^{0.35}\right)}{\partial K}\frac{\mathrm{d}K}{\mathrm{d}t} \\ &= 0.9555\left(\frac{K}{L}\right)^{0.35}(-2) + 0.5145\left(\frac{L}{K}\right)^{0.65}\cdot 0.5 \\ &= -1.911\left(\frac{8}{30}\right)^{0.35} + 0.25725\left(\frac{30}{8}\right)^{0.65} \\ &\approx -0.595832 \text{ million dollars} \\ &= -595832 \text{ dollars} \end{aligned}$$

47. Given $z = f(x-y)$.

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\mathrm{d}z}{\mathrm{d}(x-y)}\frac{\partial(x-y)}{\partial x} + \frac{\mathrm{d}z}{\mathrm{d}(x-y)}\frac{\partial(x-y)}{\partial y} = \frac{\mathrm{d}z}{\mathrm{d}(x-y)}(1-1) = 0$$

14.6 Directional Derivatives and the Gradient Vector

5. Find the directional derivative of $f(x,y) = ye^{-x}$ at $(0,4)$ in the direction indicated by the angle $\theta = 2\pi/3$.

Unit vector direction indicated by the angle $\theta = \frac{2\pi}{3}$ is $\mathbf{u} = \langle -1/2, \sqrt{3}/2 \rangle$.

$$\begin{aligned}
\mathrm{D}_{\mathbf{u}} f(0,4) &= \nabla f(0,4) \cdot \mathbf{u} \\
&= \left\langle \frac{\partial\left(ye^{-x}\right)}{\partial x}(0,4), \frac{\partial\left(ye^{-x}\right)}{\partial y}(0,4) \right\rangle \cdot \left\langle \frac{-1}{2}, \frac{\sqrt{3}}{2} \right\rangle \\
&= \left\langle \left((x,y) \mapsto -ye^{-x}\right)(0,4), \left((x,y) \mapsto e^{-x}\right)(0,4) \right\rangle \cdot \left\langle \frac{-1}{2}, \frac{\sqrt{3}}{2} \right\rangle \\
&= \langle -4, 1 \rangle \cdot \left\langle \frac{-1}{2}, \frac{\sqrt{3}}{2} \right\rangle \\
&= 2 + \frac{\sqrt{3}}{2}
\end{aligned}$$

7. Find the rate of change of $f(x,y) = \sin(2x+3y)$ at $P(-6,4)$ in the direction of the vector $\mathbf{u} = \frac{1}{2}(\sqrt{3}\hat{\mathbf{i}} - \hat{\mathbf{j}})$.

$$\begin{aligned}
\mathrm{D}_{\mathbf{u}} f(-6,4) &= \nabla f(-6,4) \cdot \mathbf{u} \\
&= \left\langle \frac{\partial \sin(2x+3y)}{\partial x}(-6,4), \frac{\partial \sin(2x+3y)}{\partial y}(-6,4) \right\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \right\rangle \\
&= \langle 2\cos(2(-6)+3\cdot 4), 3\cos(2(-6)+3\cdot 4) \rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \right\rangle \\
&= \sqrt{3} - \frac{3}{2}
\end{aligned}$$

11. Find the directional derivative of $f(x, y) = e^x \sin y$ at point $(0, \pi/3)$ in the direction of the vector $\mathbf{v} = \langle -6, 8 \rangle$

$$\begin{aligned}
\text{comp}_{\mathbf{v}} \nabla f\left(0, \frac{\pi}{3}\right) &= \frac{\nabla f\left(0, \frac{\pi}{3}\right) \cdot \mathbf{v}}{|\mathbf{v}|} \\
&= \left\langle \frac{\partial(e^x \sin y)}{\partial x}\left(0, \frac{\pi}{3}\right), \frac{\partial(e^x \sin y)}{\partial y}\left(0, \frac{\pi}{3}\right) \right\rangle \cdot \frac{\langle -6, 8 \rangle}{\sqrt{(-6)^2 + 8^2}} \\
&= \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \cdot \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle \\
&= \frac{2}{5} - \frac{3\sqrt{3}}{10}
\end{aligned}$$

17. Find the directional derivative of $h(r, s, t) = \ln(3r + 6s + 9t)$ at point $(1, 1, 1)$ in the direction of the vector $\mathbf{v} = \langle 4, 12, 6 \rangle$.

$$\begin{aligned}
\text{comp}_{\mathbf{v}} \nabla f(1, 1, 1) &= \frac{\nabla f(1, 1, 1) \cdot \mathbf{v}}{|\mathbf{v}|} \\
&= \left\langle \frac{3}{3 + 6 + 9}, \frac{6}{3 + 6 + 9}, \frac{9}{3 + 6 + 9} \right\rangle \cdot \left\langle \frac{2}{7}, \frac{6}{7}, \frac{3}{7} \right\rangle \\
&= \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle \cdot \left\langle \frac{2}{7}, \frac{6}{7}, \frac{3}{7} \right\rangle \\
&= \frac{23}{42}
\end{aligned}$$

21&25. Find the maximum rate of change of f at the given point and the direction in which it occurs.

$$f(x, y) = 4y\sqrt{x}, \qquad (4, 1) \tag{21}$$

$$\begin{aligned}
|\nabla f(4, 1)| &= \left| \left\langle \frac{\partial\left(4y\sqrt{x}\right)}{\partial x}(4, 1), \frac{\partial\left(4y\sqrt{x}\right)}{\partial y}(4, 1) \right\rangle \right| \\
&= |\langle 1, 8 \rangle| \\
&= \sqrt{65}
\end{aligned}$$

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \qquad (3, 6, -2) \tag{25}$$

$$\begin{aligned}
|\nabla f(3, 6, -2)| &= \left| \left\langle \frac{3}{\sqrt{3^2 + 6^2 + (-2)^2}}, \frac{6}{\sqrt{3^2 + 6^2 + (-2)^2}}, \frac{-2}{\sqrt{3^2 + 6^2 + (-2)^2}} \right\rangle \right| \\
&= 1
\end{aligned}$$

29. Find all points at which the direction of fastest change of the function $f(x,y) = x^2 + y^2 - 2x - 4y$ is $\hat{\mathbf{\imath}} + \hat{\mathbf{\jmath}}$.

The rate of change at point (a,b) is maximum in direction $\hat{\mathbf{\imath}} + \hat{\mathbf{\jmath}}$ if and only if $\nabla f(a,b)$ has the same direction:

$$\begin{aligned}\nabla f(a,b) \times (\hat{\mathbf{\imath}} + \hat{\mathbf{\jmath}}) = \mathbf{0} &\iff ((2x-2)\hat{\mathbf{\imath}} + (2y-4)\hat{\mathbf{\jmath}}) \times (\hat{\mathbf{\imath}} + \hat{\mathbf{\jmath}}) = 0 \\ &\iff 2(x-y+1)\hat{\mathbf{k}} = 0 \\ &\iff x - y + 1 = 0\end{aligned}$$

Thus the points satisfying given the requirement is the line whose equation is $x - y + 1 = 0$.

32. The temperature at a point (x,y,z) is given by

$$T(x,y,z) = 200e^{-x^2-3y^2-9z^2}$$

The rate of change of temperature at the point $P(2,-1,2)$ in direction $\mathbf{u}$ is

$$\begin{aligned}\mathrm{D}_{\mathbf{u}} f(2,-1,2) &= \nabla f(2,-1,2) \cdot \mathbf{u} \\ &= \left((x,y,z) \mapsto \frac{-400}{e^{x^2+3y^2+9z^2}} \langle x, 3y, 9z\rangle\right)(2,-1,2) \cdot \mathbf{u} \\ &= \frac{-400}{e^{2^2+3(-1)^2+9\cdot 2^2}} \langle 2, 3(-1), 9\cdot 2\rangle \cdot \mathbf{u} \\ &= \left\langle \frac{-800}{e^{43}}, \frac{1200}{e^{43}}, \frac{-7200}{e^{43}} \right\rangle \cdot \mathbf{u}\end{aligned}$$

For $\mathbf{u} = \langle 1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6}\rangle$, the rate of change is

$$\frac{-800}{e^{43}\sqrt{6}} + \frac{400\sqrt{6}}{e^{43}} + \frac{-1200\sqrt{6}}{e^{43}} = \frac{-10400}{e^{43}\sqrt{6}} \tag{a}$$

Temperature increases the fastest at the same direction as $\nabla f(2,-1,2)$

$$\mathbf{u} = \left\langle \frac{-2}{\sqrt{337}}, \frac{3}{\sqrt{337}}, \frac{-18}{\sqrt{337}} \right\rangle \tag{b}$$

In this direction, the rate of increase is

$$|\nabla f(2,-1,2)| = \frac{400\sqrt{337}}{e^{43}} \tag{c}$$

41. Find equations of the tangent plane and the normal line to the surface $F(x,y,z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$ at $(3,3,5)$.

Equation of the tangent plane:

$$
\begin{aligned}
&F_x(3,3,5)(x-3)+F_y(3,3,5)(y-3)+F_z(3,3,5)(z-5)=0\\
\iff\ &4(3-2)(x-3)+2(3-1)(y-3)+2(5-3)(z-5)=0\\
&\iff\ x+y+z=11
\end{aligned}
$$

Equation of the normal line:

$$
\frac{x-3}{F_x(3,3,5)}=\frac{y-3}{F_y(3,3,5)}=\frac{z-5}{F_z(3,3,5)} \iff x-3=y-3=z-5
$$

51. Given an ellipsoid

$$
E(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1
$$

Its tangent plane at the point (x_0,y_0,z_0) has the equation of

$$
\begin{aligned}
&E_x(x_0,y_0,z_0)(x-x_0)+E_y(x_0,y_0,z_0)(y-y_0)+E_z(x_0,y_0,z_0)(z-z_0)=0\\
\iff &\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0\\
\iff &\frac{2xx_0}{a^2}+\frac{2yy_0}{b^2}+\frac{2zz_0}{c^2}=\frac{2x_0^2}{a^2}+\frac{2y_0^2}{b^2}+\frac{2z_0^2}{c^2}\\
\iff &\frac{2xx_0}{a^2}+\frac{2yy_0}{b^2}+\frac{2zz_0}{c^2}=2\\
\iff &\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1
\end{aligned}
$$

56. Consider an ellipsoid $3x^2+2y^2+z^2=9$ and the sphere $x^2+y^2+z^2-8x-6y-8z+24=0$. A point in their intersection must satisfy the following equation

$$
\begin{aligned}
&x^2+y^2+z^2-8x-6y-8z+24=9-3x^2-2y^2-z^2\\
\iff &4x^2-8x+4+3y^2-6y+3+2z^2-8z+8=0\\
\iff &4(x-1)^2+3(y-1)^2+2(z-2)^2=0\\
\iff &\begin{cases} x=y=1\\ z=2\end{cases}
\end{aligned}
$$

Thus the intersection is a subset of $\{(1,1,2)\}$. Since $P(1,1,2)$ lies on both the ellipsoid and the sphere, it is the one and only intersection point of the two. Therefore, they are tangent to each other at P.

14.7 Minimum and Maximum Values

1. Suppose (1, 1) is a critical point of a function f with continuous second derivatives.

$$\begin{cases} \begin{vmatrix} f_{xx}(1,1) & f_{xy}(1,1) \\ f_{yx}(1,1) & f_{yy}(1,1) \end{vmatrix} = 4 \cdot 2 - 1^2 = 7 > 0 \\ f_{xx}(1,1) = 4 > 0 \end{cases}$$

$$\Longrightarrow f(1,1) \text{ is a local minumum} \quad \text{(a)}$$

$$\begin{vmatrix} f_{xx}(1,1) & f_{xy}(1,1) \\ f_{yx}(1,1) & f_{yy}(1,1) \end{vmatrix} = 4 \cdot 2 - 3^2 = -1 < 0$$

$$\Longrightarrow (1,1) \text{ is a saddle point of } f \quad \text{(b)}$$

7&13&15. Find the local maximum and minimum values and saddle points of the function and graph the function.

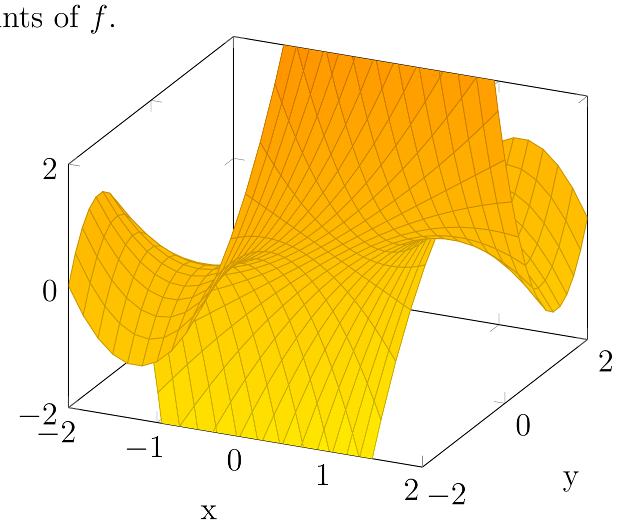
For the next few exercises, D is defined as

$$D(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix}$$

$$f(x,y) = (x-y)(1-xy) = xy^2 - x^2y + x - y \quad (7)$$

$$\begin{aligned} f_x = f_y = 0 &\iff y^2 - 2xy + 1 = 2xy - x^2 - 1 = 0 \\ &\iff x^2 = y^2 = 2xy - 1 \\ &\iff x = y = \pm 1 \end{aligned}$$

As $f_{xx} = -2y$, $f_{yy} = 2x$ and $f_{xy} = f_{yx} = 2y - 2x$, $D(x,y) = -4xy - (2y - 2x)^2$, thus $D(1,1) = D(-1,-1) = -4 < 0$. Therefore $(\pm 1, \pm 1)$ are saddle points of f.



$$f(x,y) = e^x \cos y \tag{13}$$

Since $f_x = f_y = 0 \iff e^x \cos y = -e^x \sin y = 0$ has no solution, f does not have any local minumum or maximum value.

$$f(x,y) = (x^2+y^2)e^{y^2-x^2} \tag{15}$$

$$\begin{aligned}
& f_x = f_y = 0 \\
\iff & e^{y^2-x^2}(2x + (x^2+y^2)(-2x)) = e^{y^2-x^2}(2y + (x^2+y^2)2y) = 0 \\
\iff & x^3 + xy^2 - x = x^2y + y^3 + y = 0 \\
\iff & (x^2+y^2-1)(x-y) = x^2y + y^3 + y = 0 \\
\iff & (x,y) \in \{(-1,0),(0,0),(1,0)\}
\end{aligned}$$

Second derivatives of f

$$\begin{aligned}
f_{xx} &= (4x^4 + 4x^2y^2 - 10x^2 - 2y^2 + 2)e^{y^2-x^2} \\
f_{xy} &= f_{yx} = -4xy(x^2+y^2)e^{y^2-x^2} \\
f_{yy} &= (4x^2y^2 + 4y^4 + 2x^2 + 10y^2 + 2)e^{y^2-x^2}
\end{aligned}$$

From these we can calculate $D(0,0) = 4 > 0$ and $D(\pm 1, 0) = -16/e^2 < 0$ and thus conclude that $f(0,0) = 0$ is the only local minimum value of f.

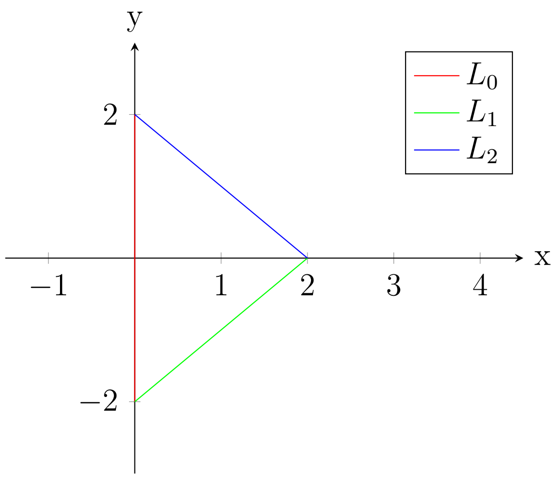
29&34. Find the absolute maximum and minimum values of f on the set D.

$$f = x^2 + y^2 - 2x, \qquad D = \{(x,y) \mid x \geq 0, |x| + |y| \leq 2\} \tag{29}$$

The critical points of f occur when

$$f_x = f_y = 0 \iff 2x - 2 = 2y = 0 \iff (x,y) = (1,0)$$

The value of f at the only critical point $(1,0)$ is $f(1,0) = 0$.



On L_0, we have $x = 0$ and

$$f(x, y) = f(0, y) = y^2, -2 \leq y \leq 2 \qquad \Longrightarrow 0 \leq f(x, y) \leq 4$$

On L_1, we have $0 \leq y = x - 2 \leq 2$ and thus

$$f(x, y) = f(x, x - 2) = 2x^2 - 6x + 4 \Longrightarrow 0 \leq f(x, y) \leq 24$$

On L_2, we have $0 \leq y = 2 - x \leq 2$ and thus

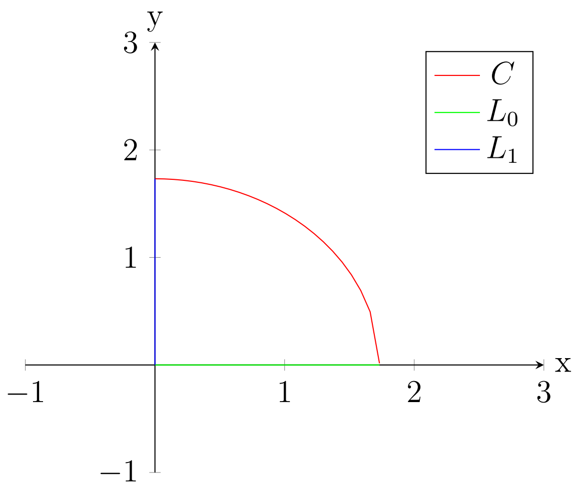
$$f(x, y) = f(x, 2 - x) = 2x^2 - 6x + 4 \Longrightarrow 0 \leq f(x, y) \leq 4$$

Therefore, on the boundary, the minimum value of f is 0 and the maximum is 24.

$$f(x, y) = xy^2, \qquad D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3\} \tag{34}$$

The critical points of f occur when

$$f_x = f_y = 0 \iff y^2 = 2xy = 0 \iff y = 0$$



The critical points of f are on L_1 and its values there are 0. On L_0, the value of $f(x, y)$ is also always 0.

On C, $y^2 = 3 - x^2$ and $0 \leq x \leq \sqrt{3}$, hence $0 \leq f(x, y) = 3x - x^3 \leq 2$.

Thus, on the boundary, the minimum value of f is 0 and the maximum is 2.

41. Find all the points $P(a, b, c)$ on the cone $z^2 = x^2 + y^2$ that are closest to the point $Q(4, 2, 0)$.

Coordinates of P satisfy $c = \sqrt{a^2 + b^2}$, thus

$$\begin{aligned} PQ^2 &= (a-4)^2 + (b-2)^2 + a^2 + b^2 \\ &= 2a^2 - 8a + 2b^2 - 4b + 20 \\ &= 2(a-2)^2 + 2(b-1)^2 + 10 \leq 10 \end{aligned}$$

Therefore the closest point to Q on the cone is $\left(2, 1, \pm\sqrt{5}\right)$. The minumum distance is $\sqrt{10}$.

49. Find the dimensions (x, y, z) of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant $c = 4(x+y+z)$.

By AM-GM inequality, the volume of the box is

$$V = xyz \leq \left(\frac{x+y+z}{3}\right)^2 = \frac{16c^2}{9}$$

Equality occurs when $x = y = z = c/12$.

14.8 Lagrange Multipliers

1. It is estimated that the minumum of f is 30 and the maximum value is 60.

5&8&13.. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given function.

$$f(x, y) = y^2 - x^2, \qquad \frac{x^2}{4} + y^2 = 1 \tag{5}$$

$$\begin{cases} \nabla f(x, y) = \lambda \nabla((x, y) \mapsto \frac{x^2}{4} + y^2) \\ \frac{x^2}{4} + y^2 = 1 \end{cases} \iff \begin{cases} \langle -2x, 2y \rangle = \lambda \left\langle \frac{x}{2}, 2y \right\rangle \\ \frac{x^2}{4} + y^2 = 1 \end{cases}$$

$$\iff \begin{cases} -2x = \frac{\lambda x}{2} \\ 2y = 2\lambda y \\ \frac{x^2}{4} + y^2 = 1 \end{cases}$$

For $x = 0$, $\lambda = 1$ and $y = \pm 1$; for $y = 0$, $\lambda = -4$ and $x = \pm 2$. Thus the minumum value of f is $f(\pm 1, 0) = -1$ and the maximum value is $f(0, \pm 2) = 4$.

$$f(x,y,z) = x^2 + y^2 + z^2, \qquad x + y + z = 12 \tag{8}$$

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla((x,y,z) \mapsto x + y + z) \\ x + y + z = 12 \end{cases} \iff \begin{cases} \langle 2x, 2y, 2z \rangle = \lambda \langle 1,1,1 \rangle \\ x + y + z = 12 \end{cases}$$

$$\iff \begin{cases} x = y = z = \frac{\lambda}{2} \\ x + y + z = 12 \end{cases}$$

$$\iff \begin{cases} x = y = z = 4 \\ \lambda = 8 \end{cases}$$

Since $f(4,4,4) = 48 < f(12,0,0) = 144$, absolute minumum value of the function subject to $x + y + z = 12$ is $f(4,4,4) = 48$.

$$f(x,y,z,t) = x + y + z + t, \qquad x^2 + y^2 + z^2 + t^2 = 1 \tag{13}$$

$$\begin{cases} \nabla f(x,y,z,t) = \lambda \nabla((x,y,z,t) \mapsto x^2 + y^2 + z^2 + t^2) \\ x^2 + y^2 + z^2 + t^2 = 1 \end{cases}$$

$$\iff \begin{cases} \langle 1,1,1,1 \rangle = \lambda \langle 2x, 2y, 2z, 2t \rangle \\ x^2 + y^2 + z^2 + t^2 = 1 \end{cases}$$

$$\iff \begin{cases} x = y = z = t = \frac{1}{2\lambda} \\ x^2 + y^2 + z^2 + t^2 = 1 \end{cases}$$

$$\iff \begin{cases} x = y = z = t = \pm\frac{1}{2} \\ \lambda = 1 \end{cases}$$

$f(-0.5,-0.5,-0.5,-0.5) = -2$ is the minumum value of f and $f(0.5,0.5,0.5,0.5) = 4$ is the maximum value.

15. Find the extreme values of $f(x, y, z) = 2x + y$ subject to $x + y + z = 1$ and $y^2 + z^2 = 4$.

Extreme values of f occur when

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla((x,y,z) \mapsto x+y+z) + \mu \nabla((x,y,z) \mapsto y^2+z^2) \\ x+y+z=1 \\ y^2+z^2=4 \end{cases}$$

$$\iff \begin{cases} \langle 2,1,0 \rangle = \lambda \langle 1,1,1 \rangle + \mu \langle 0,2y,2z \rangle \\ x+y+z=1 \\ y^2+z^2=4 \end{cases}$$

$$\iff \begin{cases} \lambda = 1 \\ \mu = \frac{1}{\sqrt{8}} \\ x = 1 \\ y = \pm\sqrt{2} \\ z = \mp\sqrt{2} \end{cases}$$

Thus the minumum value of f on the given constraints is $f(1, -\sqrt{2}) = 2 - \sqrt{2}$ and the maximum value is $f(1, \sqrt{2}) = 2 + \sqrt{2}$.

21. Find the extreme values of $f(x, y) = e^{-xy}$ on $x^2 + 4y^2 \le 1$.

Critical points of f occur when $f_x = f_y = 0 \iff x = y = 0$, the value of f there is $e^0 = 1$.

On the boundary $x^2 + 4y^2 = 1$ the minimum and maximum values can be determined using the Lagrange Method:

$$\begin{cases} \langle -ye^{-xy}, -xe^{-xy} \rangle = \lambda \langle 2x, 8y \rangle \\ x^2+4y^2=1 \end{cases} \implies \begin{cases} x \in \left\{ \frac{\pm 1}{\sqrt{2}} \right\} \\ y \in \left\{ \frac{\pm 1}{\sqrt{8}} \right\} \end{cases}$$

Thus on the boundary the minumum value of f is $e^{-1/4} = \sqrt[4]{1/e}$ and the maximum value is $\sqrt[4]{e}$. These are also the absolute extreme values of f in the ellipse.

37. Given function f on $\mathbb{R}^n_+$

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{\prod_{i=1}^{n} x_i}$$

By Lagrange Method, its extreme values subject to $\sum_{i=1}^n x_i = c$ satisfy

$$\begin{cases} \nabla f = \lambda \nabla \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i = c \end{cases} \iff \begin{cases} \left\langle \frac{x_1^{1-2/n}}{n}, \ldots, \frac{x_i^{1-2/n}}{n} \right\rangle f = \lambda \langle x_1, x_2, \ldots, x_n \rangle \\ \sum_{i=1}^n x_i = c \end{cases}$$

$$\implies \begin{cases} x_1 = x_2 = \ldots = x_n \\ \sum_{i=1}^n x_i = c \end{cases} \iff x_1 = x_2 = \ldots = x_n = \frac{c}{n}$$

At $x_1 = x_2 = \ldots = x_n = c/n$, $f(x_1, x_2, \ldots, x_n) = c/n$. As $c/n > 0 = f(c, 0, \ldots, 0)$, c/n is the maximum value of f on the given constraint.

48. By AM-GM inequality, as $\sum_{i=1}^n x_i^2 = \sum_{i=1}^n y_i^2 = 1$,

$$\sum_{i=1}^n x_i y_i \le \sum_{i=1}^n \frac{x_i^2 + y_i^2}{2} = 1$$

with equality when $\sum_{i=1}^n (x_i - y_i)^2 = 0$.

Problem Plus

1. A rectangle with length L and width W is cut into four smaller rectangles by two lines parallel to the sides.

Let x, y be two nonnegative numbers satisfying $x \le L$ and $y \le W$. The sum of the squares of the areas of the smaller rectangles would then be

$$\begin{aligned} f(x,y) &= x^2y^2 + x^2(W-y)^2 + (L-x)^2y^2 + (L-x)^2(W-y)^2 \\ &= (x^2 + (L-x)^2)(y^2 + (W-y)^2) \end{aligned}$$

By AM-GM inequality, $f(x,y) \ge 4x(L-x)y(W-y)$ with the equality $f(x,y) = L^2W^2/4$ if and only if $x = L - x = L/2$ and $y = W - y = y/2$.

On the other hand,

$$\begin{cases} 0 \le x \le L \\ 0 \le y \le W \end{cases} \implies \begin{cases} 2x(L-x) \ge 0 \\ 2y(W-y) \ge 0 \end{cases} \iff \begin{cases} L^2 \ge x^2 + (L-x)^2 \\ W^2 \ge y^2 + (W-y)^2 \end{cases}$$
$$\implies f(x,y) \le L^2W^2$$

with equality when $(x,y) \in \{(0,0), (0,W), (L,W), (L,0)\}$.

3. A long piece of galvanized sheet metal with width w is to be bent into a symmetric form with three straight sides to make a rain gutter.

Cross-section area, with $0 \le x \le w/2$ and $0 \le \theta \le \max\left(\arccos \frac{2x-w}{2x}, \pi\right)$

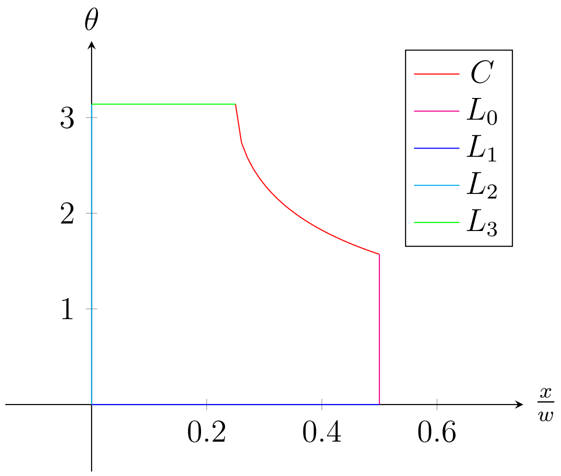
$$\begin{aligned} A(x,\theta) &= (w - 2x + x\cos\theta)x\sin\theta \\ &= wx\sin\theta - x^2\left(2\sin\theta - \frac{\sin 2\theta}{2}\right) \end{aligned}$$

First derivatives:

$$A_x = w \sin\theta - 2x\left(2\sin\theta - \frac{\sin 2\theta}{2}\right)$$
$$A_\theta = wx\cos\theta - x^2(2\cos\theta - \cos 2\theta)$$

Critical points occur when

$$A_x = A_\theta = 0 \iff \begin{cases} w\sin\theta = 2x\left(2\sin\theta - \frac{\sin 2\theta}{2}\right) \\ wx\cos\theta = x^2(2\cos\theta - \cos 2\theta) \end{cases} \qquad (*)$$



For $x = 0$ (along L_2), it is obvious that the area is 0. For $x \neq 0$,

$$(*) \iff \begin{cases} x = \frac{w\cos\theta}{2\cos\theta - \cos 2\theta} \\ w\sin\theta(2\cos\theta - \cos 2\theta) = w\cos\theta(4\sin\theta - \sin 2\theta) \end{cases}$$
$$\iff \begin{cases} x = \frac{w\cos\theta}{2\cos\theta - \cos 2\theta} \\ 2\cos\theta - \cos 2\theta = \cos\theta(4 - 2\cos\theta) \end{cases}$$
$$\iff \begin{cases} x = \frac{w\cos\theta}{2\cos\theta - \cos 2\theta} \\ -\cos 2\theta = 2\cos\theta - 2\cos^2\theta \end{cases}$$
$$\iff \begin{cases} x = \frac{w\cos\theta}{2\cos\theta - \cos 2\theta} \\ 1 = 2\cos\theta \end{cases}$$
$$\iff \begin{cases} x = \frac{w}{3} \\ \theta = \frac{\pi}{3} \end{cases}$$

At this point, $A(x, \theta) = w^2/4\sqrt{3}$.

Along C, $A\left(x, \arccos\frac{2x-w}{2x}\right) = \frac{1}{4}\sqrt{w(4x-w)(w-2x)^2} \in \left[0, \frac{w^2}{12\sqrt{3}}\right]$.

Along L_0, $A(w/2, \theta) = \frac{w^2}{8}\sin(\pi - 2\theta) \in [0, w^2/8]$.

Along L_1 and L_3, $A(x,\theta) = A(x,0) = A(x,\pi) = 0$.

In conclusion, the maximum cross-section is $\frac{w^2}{4\sqrt{3}}$ at $(x,\theta) = (w/3, \pi/3)$.

4. For what values of r is the function

$$f(x,y,z) = \begin{cases} \dfrac{(x+y+z)^r}{x^2+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$$

continuous on $\mathbb{R}^3$?

Along $y = z = 0$, as $x \to 0$, $f(x,0,0) = x^{r-2} \to \infty$ (or the limit might not exist at all) for $r < 2$ and $f(x,0,0) = 1$ for $r = 2$. Therefore for $r \leq 2$, f is discontinuous at $(0,0,0)$.

It is not difficult to show that for $r > 2$, f is continuous. For every positive number ε, let $\delta = (\varepsilon/3^r)^{1/(2r-2)}$, then from

$$\begin{aligned} & 0 < \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} < \delta \\ \iff & 0 < \sqrt{x^2+y^2+z^2} < \left(\frac{\varepsilon}{3^r}\right)^{\frac{1}{2r-2}} \\ \iff & 0 < \frac{3^r(x^2+y^2+z^2)^r}{x^2+y^2+z^2} < \varepsilon \end{aligned}$$

and

$$(x+y+z)^2 \leq 3(x^2+y^2+z^2) \iff |x+y+z|^r \leq 3^r(x^2+y^2+z^2)^r$$

we get

$$0 < \frac{|x+y+z|^r}{x^2+y^2+z^2} < \varepsilon \iff |f(x,y,z) - 0| < \varepsilon$$

Thus by definition, for $r > 2$, $f(x,y,z) \to 0$ as $(x,y,z) \to (0,0,0)$, hence f is continuous on $\mathbb{R}^3$.

5. Suppose f is a differentiable function of one variable. Show that all tangent planes to the surface $z = xf(y/x)$ intersect in a common point.

Let $t = y/x$,

$$\begin{aligned} \frac{\partial z}{\partial x} &= f(t) + x\frac{\partial f(t)}{\partial x} = f(t) + x\frac{\mathrm{d}f}{\mathrm{d}t}\frac{\partial(y/x)}{\partial x} = f(t) - t\frac{\mathrm{d}f}{\mathrm{d}t} \\ \frac{\partial z}{\partial y} &= x\frac{\partial f(t)}{\partial y} = x\frac{\mathrm{d}f}{\mathrm{d}t}\frac{\partial(y/x)}{\partial y} = \frac{\mathrm{d}f}{\mathrm{d}t} \end{aligned}$$

Equation of the tangent plane to the given surface at $P(a, b, af(b/a))$ is

$$z - af\left(\frac{b}{a}\right) = \left(f\left(\frac{b}{a}\right) - \frac{b}{a}\cdot\frac{\mathrm{d}f}{\mathrm{d}t}\left(\frac{b}{a}\right)\right)(x-a) + \frac{\mathrm{d}f}{\mathrm{d}t}\left(\frac{b}{a}\right)(y-b)$$
$$\iff z = xf\left(\frac{b}{a}\right) + \frac{\mathrm{d}f}{\mathrm{d}t}\left(\frac{b}{a}\right)\left(y - \frac{bx}{a}\right)$$
$$\iff \left(f\left(\frac{b}{a}\right) - \frac{b}{a}\cdot\frac{\mathrm{d}f}{\mathrm{d}t}\left(\frac{b}{a}\right)\right)x + \frac{\mathrm{d}f}{\mathrm{d}t}\left(\frac{b}{a}\right)y - z = 0$$

Since the equation is homogenous, the tangent plane always goes through origin $O(0, 0, 0)$.

15 Multiple Integrals

15.1 Double Integrals over Rectangles

1. Use a Riemann sum with $m = 3$ and $n = 2$ to estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle $R = [0, 6] \times [0, 4]$.

Take the sample point to be the upper right corner of each square,

$$V \approx \sum_{i=1}^{3}\sum_{j=1}^{2} ij \cdot 4 = 288 \tag{a}$$

Take the sample point to be the center of each square,

$$V \approx \sum_{i=1}^{3}\sum_{j=1}^{2}(2i-1)(2j-1)4 = 144 \tag{b}$$

13. Evaluate the double integral by first identifying it as the volume of a solid.

$$\iint_{[-2,2]\times[1,6]} (4-2y)\,\mathrm{d}A = 0$$

15.2 Integrated Integrals

Calculate the integrated integrals.

$$\int_1^4 \int_0^2 (6x^2 - 2x)\,\mathrm{d}y\,\mathrm{d}x = \int_1^4 (12x^2 - 4x)\,\mathrm{d}x = 222 \tag{3}$$

$$\int_{-3}^{3}\int_{0}^{\pi/2}(y+y^2\cos x)\,\mathrm{d}x\,\mathrm{d}y=\int_{-3}^{3}y^2\,\mathrm{d}y=0 \tag{7}$$

$$\iint_{[0,\pi/2]^2}\sin(x-y)\,\mathrm{d}A=\int_{0}^{\pi/2}(\cos y-\sin y)\,\mathrm{d}y=0 \tag{15}$$

$$\begin{aligned}\iint_{[0,1]\times[-3,3]}\frac{xy^2}{x^2+1}\,\mathrm{d}A&=\int_0^1\frac{x}{x^2+1}\,\mathrm{d}x\cdot\int_{-3}^{3}y^2\,\mathrm{d}y\\&=\frac{1}{2}\int_0^1\frac{\mathrm{d}x}{x+1}\cdot\left[\frac{y^3}{3}\right]_{-3}^{3}\\&=9\ln(x+1)\Big]_0^1\\&=9\ln 2\end{aligned} \tag{17}$$

$$\begin{aligned}\iint_{[0,2]\times[0,3]}ye^{-xy}\,\mathrm{d}A&=\int_0^3\int_0^2 ye^{-xy}\,\mathrm{d}x\,\mathrm{d}y\\&=\int_0^3(1-e^{-2y})\,\mathrm{d}y\\&=\left[y+\frac{e^{-2y}}{2}\right]_0^3\\&=\frac{1}{2e^6}+\frac{5}{2}\end{aligned} \tag{21}$$

$$\iint_{[-1,1]\times[-2,2]}\left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)\mathrm{d}A=\int_{-1}^{1}\left(\frac{92}{27}-x^2\right)\mathrm{d}x=\frac{166}{27} \tag{27}$$

$$\iint_{[0,4]\times[0,5]}(16-x^2)\,\mathrm{d}A=\int_0^4(80-5x^2)\,\mathrm{d}x=\frac{640}{3} \tag{30}$$

40. Fubini's and Clairaut's theorems are similar in the way that for continuous functions, order of variables are interchangeable in integration and differentiation. By the Fundamental Theorem and these two theorems, if $f(x,y)$ is continuous on $[a,b]\times[c,d]$ and

$$g(x,y)=\int_a^x\int_c^y g(s,t)\,\mathrm{d}t\,\mathrm{d}s$$

for $a<x<b$ and $c<y<d$, then $g_{xy}=g_{yx}=f(x,y)$.

15.3 Double Integrals over General Regions

Evaluate the iterated integral.

$$\int_0^1 \int_0^{s^2} \cos s^3 \,\mathrm{d}t\,\mathrm{d}s = \int_0^1 s^2 \cos s^3 \,\mathrm{d}s = \left[\frac{\sin s^3}{3}\right]_0^1 = \frac{\sin 1}{3} \tag{5}$$

$$\int_0^\pi \int_0^{\sin x} x \,\mathrm{d}y\,\mathrm{d}x = \int_0^\pi x \sin x \,\mathrm{d}x = [\sin x - x\cos x]_0^\pi = \pi \tag{9}$$

$$\int_{-1}^2 \int_{y^2}^{y+2} y \,\mathrm{d}x\,\mathrm{d}y = \int_{-1}^2 (2y + y^2 - y^3)\,\mathrm{d}y = \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_{-1}^2 = \frac{9}{4} \tag{15}$$

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x - y)\,\mathrm{d}y\,\mathrm{d}x = \int_{-2}^2 4x\sqrt{4-x^2}\,\mathrm{d}x = 0 \tag{21}$$

$$\begin{aligned} \int_1^2 \int_1^{7-3y} xy \,\mathrm{d}x\,\mathrm{d}y &= \int_1^2 \left(\frac{9y^3}{2} - 21y^2 + 24y\right)\mathrm{d}y \\ &= \left[\frac{9y^4}{8} - 7y^3 + 12y^2\right]_1^2 \\ &= \frac{31}{8} \end{aligned} \tag{25}$$

$$\int_1^2 \int_0^{\ln x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\ln 2} \int_{e^y}^2 f(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{47}$$

$$\int_0^1 \int_{3y}^3 e^{x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^3 \int_0^{x/3} e^{x^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^3 \frac{xe^{x^2}}{3}\,\mathrm{d}x = \left.\frac{e^{x^2}}{6}\right]_0^3 = \frac{e^9 - 1}{6} \tag{49}$$

15.4 Double Integrals in Polar Coordinates

Evaluate the given integral.

$$\int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta) r \,\mathrm{d}r\,\mathrm{d}\theta \tag{1}$$

$$\begin{aligned} \int_{\pi/4}^{\pi/2} \int_0^2 (2\cos\theta - \sin\theta) r^2 \,\mathrm{d}r\,\mathrm{d}\theta &= \int_{\pi/2}^{\pi/4} \frac{8}{3}(2\cos\theta - \sin\theta)\,\mathrm{d}\theta \\ &= \frac{8}{3}\left[2\sin\theta + \cos\theta\right]_{\pi/4}^{\pi/2} \\ &= \frac{16}{3} - 4\sqrt{2} \end{aligned} \tag{8}$$

$$\int_{-\pi/2}^{\pi/2}\int_0^2 re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \int_{-\pi/2}^{\pi/2}\frac{1-e^{-4}}{2}\,\mathrm{d}\theta$$
$$= \pi\frac{1-e^{-4}}{2} \qquad (11)$$

$$\int_0^{2\pi}\int_0^{\sqrt{1/2}}\left(\sqrt{1-r^2}-r\right) r\,\mathrm{d}r\,\mathrm{d}\theta = \pi\int_0^{\sqrt{1/2}}\left(\sqrt{1-r^2}-r\right)\,\mathrm{d}r^2$$
$$= \pi\int_0^{1/2}(\sqrt{1-x}-\sqrt{x})\,\mathrm{d}x$$
$$= \frac{\pi}{3}(2-\sqrt{2}) \qquad (25)$$

$$\int_0^{\pi}\int_0^3 r\sin r^2\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^9 \frac{\pi\sin x}{2}\,\mathrm{d}x$$
$$= \left.\frac{\pi\cos x}{-2}\right]_0^9$$
$$= \frac{\pi}{2}(1-\cos 9) \qquad (29)$$

40. We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2}\exp(-x^2-y^2)\,\mathrm{d}A$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-x^2-y^2)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \lim_{a\to\infty}\iint_{D_a}\exp(-x^2-y^2)\,\mathrm{d}A$$

where D_a is the disk with radius a and center the origin.

By changing to polar coordinates,

$$I = \lim_{a\to\infty}\int_0^{2\pi}\int_0^a \exp(-a^2)a\,\mathrm{d}a\,\mathrm{d}\theta$$
$$= \lim_{a\to\infty}\int_0^a -\pi\exp(-a^2)\,\mathrm{d}-a^2$$
$$= -\pi\lim_{a\to\infty}\int_0^{-a^2} e^b\,\mathrm{d}b$$
$$= -\pi\lim_{a\to\infty} e^b\Big]_0^{-a^2}$$
$$= \pi\lim_{a\to\infty}(1-\exp(-a^2))$$
$$= \pi \qquad \text{(a)}$$

As $\exp(-x^2 - y^2)$ is continuous on $\mathbb{R}^2$,

$$\int_{-\infty}^{\infty} \exp(-x^2)\,\mathrm{d}x \int_{-\infty}^{\infty} \exp(-y^2)\,\mathrm{d}y = I = \pi \tag{b}$$

Thus $\int_{-\infty}^{\infty} \exp(-x^2)\,\mathrm{d}x = \sqrt{I} = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} \exp(-x^2/2)\,\mathrm{d}x = \sqrt{2\pi}$.

15.5 Applications of Double Integrals

2. The total charge on the disk is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2 + y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{2\pi} \int_0^1 r^2\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi \frac{r^3}{3}\bigg]_0^1 = \frac{2\pi}{3}$$

Find the mass and center of mass of the lamina that occupies the regions D and has the given density function ρ.

$$D = [1,3] \times [1,4]; \qquad \rho(x,y) = ky^2 \tag{3}$$

$$m = \int_1^3 \mathrm{d}x \cdot \int_1^4 ky^2\,\mathrm{d}y = 42k$$

$$\bar{x} = \frac{k}{m} \int_1^3 \int_1^4 xy^2\,\mathrm{d}y\,\mathrm{d}x = \frac{21k}{m} \int_1^3 x\,\mathrm{d}x = \frac{84k}{m} = 2$$
$$\bar{y} = \frac{k}{m} \int_1^3 \int_1^4 y^3\,\mathrm{d}y\,\mathrm{d}x = \frac{2k}{m} \int_1^4 y^3\,\mathrm{d}y = \frac{255k}{m} = \frac{85}{28}$$

$$D = \{(x,y) \mid -1 \le x \le 1,\, 0 \le y \le 1 - x^2\}, \qquad \rho(x,y) = ky \tag{7}$$

$$m = \int_{-1}^{1} \int_0^{1-x^2} ky\,\mathrm{d}y\,\mathrm{d}x = \frac{k}{2} \int_{-1}^{1} (x^4 - 2x^2 + 1)\,\mathrm{d}x = \frac{8k}{15}$$

$$\bar{x} = \frac{k}{m} \int_{-1}^{1} \int_0^{1-x^2} xy\,\mathrm{d}y\,\mathrm{d}x = \frac{15}{8} \int_{-1}^{1} (x^5 - 2x^3 + x)\,\mathrm{d}x = 0$$
$$\bar{y} = \frac{k}{m} \int_{-1}^{1} \int_0^{1-x^2} y^2\,\mathrm{d}y\,\mathrm{d}x = \frac{8}{45} \int_{-1}^{1} (1 - x^2)^3\,\mathrm{d}x = \frac{4}{7}$$

$$D = \left\{(x, y) \,\middle|\, 0 \le y \le \sin\frac{\pi x}{L},\, 0 \le x \le L\right\}, \qquad \rho(x, y) = y \tag{9}$$

$$m = \int_0^L \int_0^{\sin(\pi x/L)} y \,\mathrm{d}y \,\mathrm{d}x = \int_0^L \frac{\sin^2(\pi x/L)}{2} \,\mathrm{d}x = \left[\frac{x}{4} - \frac{L}{8\pi}\sin\frac{2\pi x}{L}\right]_0^L = \frac{L}{4}$$

$$\bar{x} = \int_0^L \int_0^{\sin(\pi x/L)} \frac{xy}{m} \,\mathrm{d}y \,\mathrm{d}x = \int_0^L \frac{2x\sin^2(\pi x/L)}{L} \,\mathrm{d}x = \frac{L}{2}$$

$$\bar{y} = \int_0^L \int_0^{\sin(\pi x/L)} \frac{y^2}{m} \,\mathrm{d}y \,\mathrm{d}x = \int_0^L \frac{4\sin^3(\pi x/L)}{3L} \,\mathrm{d}x = \frac{16}{9\pi}$$

$$D = \{(x, y) \,|\, 0 \le x \le 1,\, 0 \le y \le \sqrt{1 - x^2}\}, \qquad \rho(x, y) = ky \tag{11}$$

$$m = \int_0^1 \int_0^{\sqrt{1-x^2}} ky \,\mathrm{d}y \,\mathrm{d}x = \int_0^{\pi/2} \sin\theta \,\mathrm{d}\theta \cdot \int_0^1 kr^2 \,\mathrm{d}r = \frac{k}{3}$$

$$\bar{x} = \int_0^1 \int_0^{\sqrt{1-x^2}} 3xy \,\mathrm{d}y \,\mathrm{d}x = \int_0^{\pi/2} \cos\theta \sin\theta \,\mathrm{d}\theta \cdot \int_0^1 3r^3 \,\mathrm{d}r = \frac{3}{8}$$

$$\bar{y} = \int_0^1 \int_0^{\sqrt{1-x^2}} 3y^2 \,\mathrm{d}y \,\mathrm{d}x = \int_0^{\pi/2} \sin^2\theta \,\mathrm{d}\theta \cdot \int_0^1 3r^3 \,\mathrm{d}r = \frac{3\pi}{16}$$

15.6 Surface area

Find the area of the surface.
3. The part of the plane $3x + 2y + z = 6$ that lies in the first octant.

$$\begin{aligned}
A &= \int_0^2 \int_0^{3-1.5x} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_0^2 \int_0^{3-1.5x} \sqrt{14} \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_0^2 \left(3 - \frac{3}{2}x\right) \sqrt{14} \,\mathrm{d}x \\
&= \left[3x\sqrt{14} - \frac{3x^2\sqrt{14}}{4}\right]_0^2 \\
&= 3\sqrt{14}
\end{aligned}$$

9. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

$$\begin{aligned}
A &= \iint_D \sqrt{1 + \left(\frac{\partial xy}{\partial x}\right)^2 + \left(\frac{\partial xy}{\partial y}\right)^2}\,\mathrm{d}A \\
&= \int_0^{2\pi}\int_0^1 r\sqrt{1+r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \pi\int_0^1 \sqrt{1+t}\,\mathrm{d}t \\
&= \left.\frac{2\pi\sqrt{(1-t)^3}}{3}\right]_0^1 \\
&= \frac{2\pi}{3}\left(2\sqrt{2}-1\right)
\end{aligned}$$

12. The part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$, in which it has the equation $z = 2 + \sqrt{4 - x^2 - y^2}$.

$$\begin{aligned}
A &= \iint_D \sqrt{1 + \left(\frac{\partial}{\partial x}\left(2+\sqrt{4-x^2-y^2}\right)\right)^2 + \left(\frac{\partial}{\partial y}\left(2+\sqrt{4-x^2-y^2}\right)\right)^2}\,\mathrm{d}A \\
&= \iint_D \sqrt{\frac{4}{4-x^2-y^2}}\,\mathrm{d}A \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} r\sqrt{\frac{4}{4-r^2}}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\pi\int_0^3 \sqrt{\frac{1}{4-t}}\,\mathrm{d}t \\
&= \left.-4\pi\sqrt{4-t}\right]_0^3 \\
&= 4\pi
\end{aligned}$$

15.7 Triple Integrals

Evaluate the integral.

$$\int_0^1\int_0^3\int_{-1}^2 xyz^2\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x = \int_0^1\int_0^3 \frac{3xz^2}{2}\,\mathrm{d}z\,\mathrm{d}x = \int_0^1 \frac{27x}{2}\,\mathrm{d}x = \frac{27}{4} \tag{1}$$

$$\begin{aligned}\int_0^2\int_0^{z^2}\int_0^{y-z}(2x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^2\int_0^{z^2}(z^2-yz)\,\mathrm{d}y\,\mathrm{d}z\\ &= \int_0^2\left(z^4-\frac{z^5}{2}\right)\,\mathrm{d}z\\ &= \frac{16}{15}\end{aligned} \tag{3}$$

$$\int_0^3\int_0^x\int_{x-y}^{x+y} y\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^3\int_0^x 2y^2\,\mathrm{d}y\,\mathrm{d}x = \int_0^3 \frac{2x^3}{3}\,\mathrm{d}x = \frac{27}{2} \tag{9}$$

$$\begin{aligned}\int_0^{\pi}\int_0^{\pi-x}\int_0^x \sin y\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{\pi}\int_0^{\pi-x} x\sin y\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^{\pi}(x+x\cos y)\,\mathrm{d}x\\ &= \frac{\pi^2}{2}-2\end{aligned} \tag{12}$$

$$\begin{aligned}\int_0^1\int_0^{3x}\int_0^{\sqrt{9-y^2}} z\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1\int_0^{3x}\frac{9-y^2}{2}\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^1\frac{27x-9x^3}{2}\,\mathrm{d}x\\ &= \frac{45}{8}\end{aligned} \tag{18}$$

$$\begin{aligned}\int_0^2\int_0^{4-2x}\int_0^{4-2x-y}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x &= \int_0^2\int_0^{4-2x}(4-2x-y)\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^2\frac{(4-2x)^2}{2}\,\mathrm{d}x\\ &= \frac{16}{3}\end{aligned} \tag{19}$$

$$\begin{aligned}\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-1}^{4-z}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x &= \int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(5-z)\,\mathrm{d}z\,\mathrm{d}x\\ &= \int_{-2}^2 10\sqrt{4-x^2}\,\mathrm{d}x\\ &= 20\pi\end{aligned} \tag{22}$$

15.8 Triple Integrals in Cylindrical Coordinates

1. Change from cylindrical coordinates to rectangular coordinates.

(a) $\left(4, \frac{\pi}{3}, -2\right) \to \left(2, 2\sqrt{3}, -2\right)$

(b) $\left(2, \frac{-\pi}{2}, 1\right) \to (0, -2, 1)$

3. Change from rectangular coordinates to cylindrical coordinates.

(a) $(-1, 1, 1) \to \left(\sqrt{2}, \frac{3\pi}{4}, 1\right)$

(b) $\left(-2, 2\sqrt{3}, 3\right) \to \left(4, \frac{2\pi}{3}, 3\right)$

7. In cylindrical coordinates (r, θ, z), $z = 4 - r^2$ is the paraboloid $z = 4 - x^2 - y^2$ in Cartesian coordinates.
15&17&21. Evaluate the integral.

$$\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{r^2} r \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\theta = \pi \int_0^2 r^3 \,\mathrm{d}r = 4\pi \tag{15}$$

$$\iiint_E \sqrt{x^2 + y^2} \,\mathrm{d}V = \int_0^{2\pi} \int_0^4 \int_{-5}^4 r^2 \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\theta = 18\pi \left.\frac{r^3}{3}\right]_0^4 = 384\pi \tag{17}$$

$$\begin{aligned} \iiint_E x^2 \,\mathrm{d}V &= \int_0^{2\pi} \int_0^2 \int_{z/2}^1 r^3 \cos^2\theta \,\mathrm{d}r \,\mathrm{d}z \,\mathrm{d}\theta \\ &= \int_0^{2\pi} \cos^2\theta \,\mathrm{d}\theta \int_0^2 \int_{z/2}^1 r^3 \,\mathrm{d}r \,\mathrm{d}z \\ &= \left[\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}\right]_0^{2\pi} \int_0^2 \left(\frac{1}{4} - \frac{z^4}{64}\right) \mathrm{d}z \\ &= \frac{2\pi}{5} \end{aligned} \tag{21}$$

16 Vector Calculus

16.2 Line Integrals

Evaluate the integral.

$$\begin{aligned} &\int_{-\pi/2}^{\pi/2} 4\cos t (4\sin t)^4 \sqrt{\left(\frac{\mathrm{d}4\cos t}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}4\sin t}{\mathrm{d}t}\right)^2} \,\mathrm{d}t \\ = &4096 \int_{-\pi/2}^{\pi/2} \sin^4 t \,\mathrm{d}\sin t = 4096 \int_{-1}^1 w^4 \,\mathrm{d}w = \frac{8192}{5} \end{aligned} \tag{3}$$

$$\begin{aligned}\int_{\left\{(x,y)\in[1,4]\times[1,2]\,|\,y=\sqrt{x}\right\}} \left(x^2y^3-\sqrt{x}\right)\,\mathrm{d}y &= \int_1^2 (t^7-t)\frac{\mathrm{d}t}{\mathrm{d}t}\,\mathrm{d}t \\ &= \left[\frac{t^8}{8}-\frac{t^2}{2}\right]_1^2 \\ &= \frac{243}{8} \end{aligned} \tag{5}$$

$$\begin{aligned}&\int_0^2 (x+x)\,\mathrm{d}x + \int_2^3 (x+6-2x)\,\mathrm{d}x + \int_0^1 (2y)^2\,\mathrm{d}y + \int_1^0 (3-x)^2\,\mathrm{d}y \\ &= 4+\frac{7}{2}+\frac{4}{3}-\frac{19}{3}=\frac{5}{2}\end{aligned} \tag{7}$$

$$\begin{aligned}&\int_2^0 x^2\,\mathrm{d}x + \int_0^4 x^2\,\mathrm{d}x + \int_0^2 y^2\,\mathrm{d}y + \int_2^3 \,\mathrm{d}y \\ &= \int_2^4 x^2\,\mathrm{d}x + \int_0^3 y^2\,\mathrm{d}y = \left.\frac{x^3}{3}\right]_2^4 + \left.\frac{y^3}{3}\right]_0^3 = 13\end{aligned} \tag{8}$$

$$\begin{aligned}\int_0^1 (11y^7\hat{\mathbf{i}}+3t^6\hat{\mathbf{j}})\,\mathrm{d}(11t^4\hat{\mathbf{i}}+t^3\hat{\mathbf{j}}) &= \int_0^1 (11y^7\hat{\mathbf{i}}+3t^6\hat{\mathbf{j}})\cdot(44t^3\hat{\mathbf{i}}+3t^2\hat{\mathbf{j}})\,\mathrm{d}t \\ &= \int_0^1 (484t^{10}+9t^8)\,\mathrm{d}t \\ &= \left[44t^11+t^9\right]_0^1 \\ &= 45\end{aligned} \tag{19}$$

$$\begin{aligned}&\int_0^1 (\sin t^3\hat{\mathbf{i}}+\cos t^2\hat{\mathbf{j}}+t^4\hat{\mathbf{k}})\,\mathrm{d}(t^3\hat{\mathbf{i}}+t^2\hat{\mathbf{j}}+t\hat{\mathbf{k}}) \\ &= \int_0^1 \sin x\,\mathrm{d}x + \int_0^1 \cos y\,\mathrm{d}y + \int_0^1 z^4\,\mathrm{d}z \\ &= \frac{6}{5}-\cos 1-\sin 1\end{aligned} \tag{21}$$

$$
\begin{aligned}
&\int_0^{2\pi} (t - \sin t)\,\mathrm{d}(t - \sin t) + (3 - \cos t)\,\mathrm{d}(1 - \cos t) \\
&= \int_0^{2\pi} ((t - \sin t)(1 - \cos t) + (3 - \cos t)\sin t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (t - t\cos t + 2\sin t)\,\mathrm{d}t \\
&= \left[\frac{t^2}{2} - t\sin t - 3\cos t\right]_0^{2\pi} = 2\pi^2
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&2\int_0^{2\pi} \left(4 + \frac{x^2 - y^2}{100}\right) \sqrt{(-10\sin t)^2 + (10\cos t)^2}\,\mathrm{d}t \\
&= \int_0^{2\pi} \left(800 + (10\cos t)^2 - (10\sin t)^2\right)\,\mathrm{d}t \\
&= 100\int_0^{2\pi} (8 + \cos 2t)\,\mathrm{d}t \\
&= \left[8t - \frac{\sin 2t}{2}\right]_0^{2\pi} = 16\pi
\end{aligned} \tag{48}
$$

16.3 The Fundamental Theorem for Line Integrals

Evaluate the integrals.

$$
\begin{aligned}
&\int_C (x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}}) \cdot \mathrm{d}(x\hat{\mathbf{i}} + 2x^2\hat{\mathbf{j}}) \\
&= (f \mapsto f(2,8) - f(-1,2))\left((x,y) \mapsto \frac{x^3 + y^3}{3}\right) = 513
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&\int_C (xy^2\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}}) \cdot \mathrm{d}\mathbf{r} \\
&= (f \mapsto f(2,1) - f(0,1))\left((x,y) \mapsto \frac{x^2y^2}{2}\right) = 2
\end{aligned} \tag{13}
$$

16.4 Green's Theorem

Evaluate the integrals.

$$\begin{aligned}
\int_C \left(y + e^{\sqrt{x}}\right) \mathrm{d}x + (2x + \cos y^2) \,\mathrm{d}y &= \int_0^1 \int_{y^2}^{\sqrt{y}} \mathrm{d}x \,\mathrm{d}y \\
&= \int_0^1 (\sqrt{y} - y^2) \,\mathrm{d}y \\
&= \left[\frac{2\sqrt{y^3}}{3} - \frac{y^3}{3} \right]_0^1 \\
&= \frac{1}{3} \qquad (7)
\end{aligned}$$

$$\begin{aligned}
\int_{x^2+y^2=4} y^3 \,\mathrm{d}x - x^3 \,\mathrm{d}y &= \iint_{x^2+y^2=4} (-3x^2 - 3y^2) \,\mathrm{d}A \\
&= -3 \int_0^{2\pi} \int_0^2 r^3 \,\mathrm{d}r \,\mathrm{d}\theta \\
&= -6\pi \left. \frac{r^4}{4} \right]_0^2 \\
&= -24\pi \qquad (9)
\end{aligned}$$

$$\begin{aligned}
\int_C (1 - y^3) \,\mathrm{d}x + (x^3 + \exp y^2) \,\mathrm{d}y &= \iint_D (3x^2 + 3y^2) \,\mathrm{d}A \\
&= 3 \int_0^{2\pi} \int_2^3 r^3 \,\mathrm{d}r \,\mathrm{d}\theta \\
&= 6\pi \left. \frac{r^4}{4} \right]_2^3 \\
&= \frac{195}{8}\pi \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&\int_C (y \cos x - xy \sin x) \,\mathrm{d}x + (xy + x \cos x) \,\mathrm{d}y \\
&= -\iint_D (y + \cos x - x \sin x - \cos x + x \sin x) \,\mathrm{d}A \\
&= -\int_0^2 \int_0^{4-2x} y \,\mathrm{d}y \,\mathrm{d}x = \frac{16}{-3} \qquad (11)
\end{aligned}$$

$$\begin{aligned}
&\int_C (\exp -x + y^2)\,\mathrm{d}x + (\exp -y + x^2)\,\mathrm{d}y \\
&= -\int_{-\pi/2}^{\pi/2}\int_0^{\cos x} (2x - 2y)\,\mathrm{d}y\,\mathrm{d}x \\
&= -\int_{-\pi/2}^{\pi/2} (2x\cos x - \cos^2 x)\,\mathrm{d}x = \frac{\pi}{2} \qquad (12)
\end{aligned}$$

$$\int_0^1\int_0^{1-x} (y^2 - x)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left(\frac{(1-x)^3}{3} + x^2 - x\right)\mathrm{d}x = \frac{-1}{12} \qquad (17)$$

$$\begin{aligned}
\int_{\text{cycloid}} y\,\mathrm{d}x + \int_{\text{segment}} y\,\mathrm{d}x &= \int_{2\pi}^0 (1-\cos t)\,\mathrm{d}(t - \sin t) + 0 \\
&= \int_{2\pi}^0 \left(\frac{3}{2} - 2\cos t + \frac{\cos 2t}{2}\right)\mathrm{d}t \\
&= \left[\frac{3t}{2} - 2\sin t + \frac{\sin 2t}{4}\right]_{2\pi}^0 = 3\pi \qquad (19)
\end{aligned}$$

16.5 Curl and Divergence

19. Since the divergence of curl of $\mathbf{G}$ is $1 \neq 0$, there does not exist a vector field $\mathbf{G}$ satisfying the given condition.

16.6 Parametric Surfaces and Their Areas

19. One parametric representation for the surface $x + y + z = 0$ is $\mathbf{r}(u, v) = \langle u, v, -u - v\rangle$.
23. One parametric representation for the sphere $x^2 + y^2 + z^2 = 4$ above the cone $\sqrt{x^2 + y^2}$ is $\mathbf{r}(u, v) = \langle 2\cos u\cos v, 2\cos u\sin v, 2\sin u\rangle$.
39. The plane intersects with Ox at $A(2, 0, 0)$, with Oy at $B(0, 3, 0)$ and with Oz at $C(0, 0, 6)$. The area of the triangle ABC is $|\mathbf{AB} \times \mathbf{AC}|/2 = 3\sqrt{14}$.
42. Surface of the cone $\sqrt{x^2 + y^2}$:

$$\iint_D \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2}\,\mathrm{d}A = \iint_D \sqrt{2}\,\mathrm{d}A$$

For the part lying between $y = x$ and $y = x^2$, the area is

$$\int_0^1\int_{x^2}^x \sqrt{2}\,\mathrm{d}y\,\mathrm{d}x = \sqrt{2}\int_0^1 (x - x^2)\,\mathrm{d}x = \frac{\sqrt{2}}{6}$$

43. Area of the surface:

$$\int_0^1 \int_0^1 \sqrt{1+x+y}\,\mathrm{d}y\,\mathrm{d}x = \frac{4-32\sqrt{2}}{15} + \frac{12\sqrt{3}}{5}$$

45. Area of $z = xy$ within $x^2+y^2=1$:

$$\iint_D \sqrt{1+x^2+y^2}\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}r\,\mathrm{d}r\,\mathrm{d}\theta = \pi\int_1^2 \sqrt{t}\,\mathrm{d}t = \frac{2\pi}{3}\left(\sqrt{8}-1\right)$$

49. Area of the surface with given parametric equation $\mathbf{r}(u,v) = \langle u^2, uv, v^2/2\rangle$ within $0 \le u \le 1$ and $0 \le v \le 2$:

$$\iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}A = \int_0^2\int_0^1 (2u^2+v^2)\,\mathrm{d}u\,\mathrm{d}v = \int_0^2 \left(\frac{2}{3}+v^2\right)\,\mathrm{d}v = 4$$

16.7 Surface Integrals

Evaluate the surface integrals.

$$\begin{aligned}\iint_S (x+y+z)\,\mathrm{d}S &= \int_0^2\int_0^1 (4u+v+1)\sqrt{14}\,\mathrm{d}v\,\mathrm{d}u \\ &= \int_0^2 \left(4u+\frac{3}{2}\right)\sqrt{14}\,\mathrm{d}u \\ &= 11\sqrt{14} \end{aligned} \qquad (5)$$

$$\begin{aligned}\int_0^2\int_0^3 x^2y(1+2x+3y)\sqrt{1+4+9}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^2 \left(27y^2+\frac{99}{2}y\right)\sqrt{14}\,\mathrm{d}y \\ &= 171\sqrt{14}\end{aligned} \qquad (9)$$

$$\begin{aligned}&\int_0^1\int_0^1 \left(xy\hat{\mathbf{i}}+yz\hat{\mathbf{j}}+zx\hat{\mathbf{k}}\right)\cdot\left(\hat{\mathbf{i}}+0\hat{\mathbf{j}}-2x\hat{\mathbf{k}}\right)\times\left(0\hat{\mathbf{i}}+\hat{\mathbf{j}}-2y\hat{\mathbf{k}}\right)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1\int_0^1 \left(xz+2y^2z+2x^2y\right)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1\int_0^1 ((x+2y^2)(4-x^2-y^2)+2x^2y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1\int_0^1 (4x-x^3-xy^2+8y^2-2x^2y^2-2y^4+2x^2y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \left(4x-x^3-\frac{x}{3}+\frac{8}{3}-\frac{2x^2}{3}-\frac{2}{5}+x^2\right)\,\mathrm{d}x \\ &= 2-\frac{1}{4}-\frac{1}{6}+\frac{8}{3}-\frac{2}{9}-\frac{2}{5}+\frac{1}{3} = \frac{713}{180}\end{aligned} \qquad (23)$$

17 Second-Order Linear Equations

17.1 Homogeneous Linear Equations

Solve the differential equation.

$$y'' - y' - 6y = 0 \tag{1}$$

The auxiliary equation is $r^2 - r - 6 = 0$ whose roots are $r = -2, 3$. Therefore, the general solution of the given differential equation is

$$y = \frac{c_1}{e^{2x}} + c_2 e^{3x}$$

$$y'' + 16y = 0 \tag{3}$$

The auxiliary equation is $r^2 + 16 = 0$ whose roots are $r = \pm 4i$. Therefore, the general solution of the given differential equation is

$$y = c_1 \cos 4x + c_2 \sin 4x$$

$$9y'' - 12y' + 4y = 0 \tag{5}$$

The auxiliary equation is $9r^2 - 12r + 4 = 0$ whose roots are $r_1 = r_2 = 2/3$. Therefore, the general solution of the given differential equation is

$$y = (c_1 + c_2 x)e^{2x/3}$$

$$2y'' = y' \tag{7}$$

The auxiliary equation is $2r^2 = r$ whose roots are $r = 0, 1/2$. Therefore, the general solution of the given differential equation is $y = c_1 + c_2\sqrt{e^x}$.

$$y'' - 6y' + 8y = 0, \qquad y(0) = 2, \qquad y'(0) = 2 \tag{17}$$

The auxiliary equation is $r^2 - 6r + 8 = 0$ whose roots are $r = 2, 4$. Therefore, the general solution of the given differential equation is

$$y = c_1 e^{2x} + c_2 e^{4x} \Longrightarrow y' = 2c_1 e^{2x} + 4c_2 e^{4x}$$

Since $y(0) = y'(0) = 2$,

$$c_1 + c_2 = 2c_1 + 4c_2 = 2 \iff (c_1, c_2) = (3, -1) \iff y = 3e^{2x} - e^{4x}$$

$$9y'' + 12y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 0 \tag{19}$$

The auxiliary equation is $9r^2+12r+4 = 0$ whose roots are $r_1 = r_2 = -2/3$. Therefore, the general solution of the given differential equation is

$$y = \frac{c_1 + c_2x}{e^{2x/3}} \Longrightarrow y' = \frac{c_2 - 2c_2x/3 - 2c_1/3}{e^{2x/3}}$$

As $y(0) = 1$, $c_1 = 1$ and as $y'(0) = 0$, $c_2 = 2/3$, thus

$$y = \left(1 + \frac{2x}{3}\right) e^{-2x/3}$$

17.2 Nonhomogeneous Linear Equations

Solve the differential equation.

$$y'' - 2y' - 3y = \cos 2x \tag{1}$$

The auxiliary equation of $y'' - 2y' - 3y = 0$ is $r^2 - 2r - 3 = 0$ with roots $r = -1, 3$. So the solution of the complementary equation is

$$y_c = \frac{c_1}{e^x} + c_2e^{3x}$$

Since $G(x) = \cos 2x$ is cosine function, we seek a particular solution of the form $y_p = A\sin 2x + B\cos 2x$. Then $y_p' = 2A\cos 2x - 2B\sin 2x$ and $y_p'' = -4y$ so, substituting into the given differential equation, we have

$$(4A - 7B)\cos 2x - (7A + 4B)\sin 2x = \cos 2x$$

$$\Longleftrightarrow \begin{cases} 4A - 7B = 1 \\ 7A + 4B = 0 \end{cases} \Longleftrightarrow \begin{cases} A = \dfrac{4}{65} \\ B = \dfrac{-7}{65} \end{cases}$$

Thus the general solution of the given differential equation is

$$y = y_c + y_p = \frac{c_1}{e^x} + c_2e^{3x} + \frac{4\sin 2x}{65} - \frac{7\cos 2x}{65}$$

$$y'' + 9y = \frac{1}{e^{2x}} \tag{3}$$

The auxiliary equation of $y'' + 9y = 0$ is $r^2 + 9 = 0$ whose roots are $r = \pm 3i$. Therefore, the general solution of the given differential equation is

$$y_c = c_1\cos 3x + c_2\sin 3x$$

Since $G(x) = e^{-2x}$ is an exponential function, we seek a particular solution of an exponential function as well:

$$y_p = Ae^{-2x} \Longrightarrow y_p' = -2Ae^{-2x} \Longrightarrow y_p'' = 4Ae^{-2x}$$

Substituting these into the differential equation, we get

$$\frac{13A}{e^{2x}} = \frac{1}{e^{2x}} \iff A = \frac{1}{13} \iff y_p = \frac{1}{13e^{2x}}$$

Thus the general solution of the given differential equation is

$$y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{13e^{2x}}$$

$$y'' - 4y = e^x \cos x, \qquad y(0) = 1, \qquad y'(0) = 2 \tag{8}$$

The auxiliary equation of $y'' + 4y = 0$ is $r^2 + 4 = 0$ whose roots are $r = \pm 2i$. Therefore, the general solution of the given differential equation is

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

We seek a particular solution of the form $y_p = e^x(A \sin x + B \cos x)$. Substituting this into the given differential equation we get

$$2e^x(A\cos x - B\sin x) + 4e^x(A\sin x + B\cos x) = e^x \cos x$$
$$\iff \begin{cases} 2A + 4B = 1 \\ 4A - 2B = 0 \end{cases} \iff \begin{cases} A = 0.1 \\ B = 0.2 \end{cases}$$

Thus the general solution of the given differential equation is

$$y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + e^x(0.1 \sin x + 0.2 \cos x)$$
$$\Longrightarrow y' = 2c_2 \cos 2x - 2c_1 \sin 2x + e^x(0.3 \cos x - 0.1 \sin x)$$

From $y(0) = 1$ we obtain $c_1 = 0.8$ and from $y'(0) = 2$ we have $c_2 = 0.85$. Thus the solution of the initial-value problem is

$$y = 0.8 \cos 2x + 0.85 \sin 2x + e^x(0.1 \sin x + 0.2 \cos x)$$

$$y'' - y' = xe^x, \qquad y(0) = 2, \qquad y'(0) = 1 \tag{9}$$

The auxiliary equation of $y'' - y' = 0$ is $r^2 - r = 0$ with roots $r = 0, 1$. So the solution of the complementary equation is

$$y_c = c_1 + c_2 e^x$$

Base on instinct, we seek a particular solution of the form $y_p = (A+x)e^x$. Substituting this into the given differential equation we get

$$(2 + A + x)e^x + (1 + A + x)e^x = xe^x \iff 3 + 2A = 0 \iff A = \frac{-3}{2}$$

Thus the general solution of the given differential equation is

$$y = y_c + y_p = c_1 + c_2e^x + \left(x - \frac{3}{2}\right) e^x = c_1 + (x + C)e^x$$
$$\implies y' = (x + C + 1)e^x$$

From $y'(0) = 1$ we get $C = 0$ and from $y(0) = 2$ we get $c_1 = 2$. Hence the solution of the initial-value problem is $y = c_1 + (x + C)e^x$.