Cuculutu Review

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14 Partial Derivatives

14.2 Limits et Continuity

37. Determine the set of points at which the function is continuous.

$$f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

By AM-GM inequality,

$$x^2+x^2+y^2 \geq 3x^2|y| \iff \frac{x^2|y^3|}{3x^2|y|} \geq \frac{x^2|y^3|}{2x^2+y^2} \geq 0 \iff \frac{-y^2}{3} \leq \frac{x^2y^3}{2x^2+y^2} \leq \frac{y^2}{3}$$

Since $\pm y^2/3 \to 0$ as $y \to 0$, by the Squeeze Theorem,

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 \neq f(0,0)$$

Therefore f is discontiuous at (0, 0). On $\mathbb{R}^2\backslash\{0\}$, f is a rational function and thus is continuous on its domain.

44. Let

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \geq x^4 \\ 1 & \text{if } 0 < y < x^4 \end{cases}$$

(a) For all paths of the form $y = mx^a$ with $a < 4 \iff 4 - a > 0$, consider the function $g(x) = |y| - x^4 = |m| \cdot |x|^a - |x|^4$:

$$g(x) \geq 0 \iff |m| \cdot |x|^a \geq |x|^4 \iff |x| \leq \sqrt[4-a]{|m|}$$

When this condition is met, either $y \le 0$ or $y = |y| \ge x^4$, so $f(x, y) = 0$. Therefore $f(x, y) = 0 \to 0$ as $(x, y) \to (0, 0)$ on

$$\left\{(x, y) \,\middle|\, x \in \left[-\sqrt[4-a]{|m|},\ \sqrt[4-a]{|m|}\right] \cap D\right\}$$

which includes the point (0, 0) if the domain D of $x \mapsto mx^a$ does.

(b) It is trivial that $f(0, 0) = 0$. Along $y = x^4/2$, for $x \neq 0$,

$$x^4 - y = x^4 - \frac{x^4}{2} = \frac{x^4}{2} > 0 \iff y < x^4 \implies f(x, y) = 1$$

Hence

$$\lim_{\substack{x\to 0\\ y\to 0}} f\left(x, \frac{x^4}{2}\right) = 1 \neq f(0, 0) = 0$$

or f is discontiuous on $y = x^4/2$ at (0, 0).

(c) Using the same reasoning, one may also easily show that f is discontiuous on the entire curve $y = x^4/20$.

14.3 Partial Derivatives

33. Find the first partial derivatives of the function.

$$\begin{aligned} w &= \ln(x + 2y + 3z) \\ \frac{\partial w}{\partial x} &= \frac{1}{x + 2y + 3z} \cdot \frac{\partial(x + 2y + 3z)}{\partial x} = \frac{1}{x + 2y + 3z} \\ \frac{\partial w}{\partial y} &= \frac{1}{x + 2y + 3z} \cdot \frac{\partial(x + 2y + 3z)}{\partial y} = \frac{2}{x + 2y + 3z} \\ \frac{\partial w}{\partial z} &= \frac{1}{x + 2y + 3z} \cdot \frac{\partial(x + 2y + 3z)}{\partial z} = \frac{3}{x + 2y + 3z} \end{aligned}$$

50. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

$$yz + x \ln y = z^2 \implies \begin{cases} y\dfrac{\partial z}{\partial x} + \ln y & = 2z\dfrac{\partial z}{\partial x} \\ z + \dfrac{x}{y} & = 2z\dfrac{\partial z}{\partial y} \end{cases} \iff \begin{cases} \dfrac{\ln y}{2z - y} & = \dfrac{\partial z}{\partial x} \\ 2 + \dfrac{x}{2yz} & = \dfrac{\partial z}{\partial y} \end{cases}$$

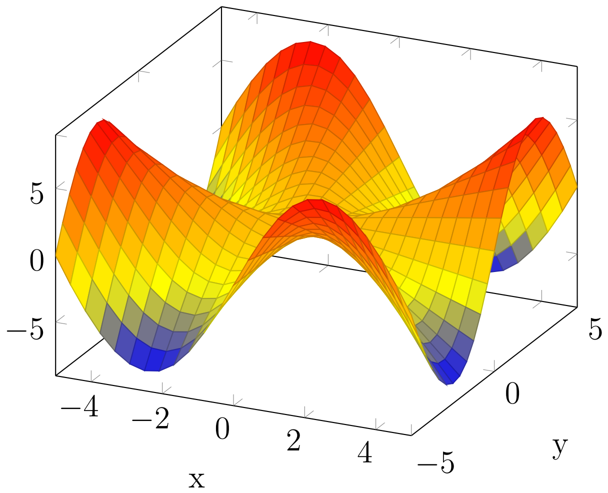
66. Find g_{rst}.

$$\begin{aligned} g(r, s, t) = e^r \sin(st) &\implies g_r = e^r \sin(st) \\ &\implies g_{rs} = se^r \cos(st) \implies g_{rst} = -ste^r \sin(st) \end{aligned}$$

101. Let

$$f(x,y) = \begin{cases} \dfrac{x^3y + xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Graph f.



(b) Find the first partial derivatives of f when $(x,y) \neq (0,0)$.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{(x^2+y^2)\dfrac{\partial(x^3y - xy^3)}{\partial x} - (x^3y - xy^3)\dfrac{\partial(x^2+y^2)}{\partial x}}{(x^2+y^2)^2} \\
&= \frac{(x^2+y^2)(3x^2y - y^3) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} \\
&= \frac{x^4y + 4x^2y^3 - y^5}{x^4 + 2x^2y^2 + y^4}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{(x^2+y^2)(x^3 - 3xy^2) - 2y(x^3y - xy^3)}{(x^2+y^2)^2} \\
&= \frac{x^5 - 4x^3y^2 - xy^4}{x^4 + 2x^2y^2 + y^4}
\end{aligned}$$

(c) Find f_x, f_y at (0, 0).

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3 0 - h0^3}{h^2+0^2} - 0}{h} = \lim_{h\to 0} 0 = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0$$

(d) Show that $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0+0-h^5}{0+0+h^4} - 0}{h} = \lim_{h\to 0} -1 = -1$$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^5+0+0}{h^4+0+0} - 0}{h} = \lim_{h\to 0} 1 = 1$$

(e) The result of part (d) does not contradict Clairaut's Theorem, which only covers the case f_{xy} and f_{yx} are continuous at (0, 0). Using GeoGebra we get the second derivatives of f on $\mathbb{R}\backslash\{0\}$ as followed:

$$f_{xy} = f_{yx} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Since $f_{xy}(x,0) = x^6/x^6 \to 1$ while $f_{xy} = -y^6/y^6 \to -1$ as $(x,y) \to (0,0)$ the second derivative is discontinuous at origin.

14.6 Directional Derivatives

17. Find the directional derivative of $h(r,s,t) = \ln(3r+6s+9t)$ at (1, 1, 1) in the direction of $\mathbf{v} = 4\hat{\mathbf{i}} + 12\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$.

From gradient of h

$$\nabla h = \frac{3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 9\hat{\mathbf{k}}}{3r+6s+9t} \Longrightarrow \nabla h(1,1,1) = \frac{\hat{\mathbf{i}}}{6} + \frac{\hat{\mathbf{j}}}{3} + \frac{\hat{\mathbf{k}}}{2}$$

and unit vector of $\mathbf{v}$

$$\hat{\mathbf{v}} = \frac{2\hat{\mathbf{i}}}{7} + \frac{6\hat{\mathbf{j}}}{7} + \frac{3\hat{\mathbf{k}}}{7}$$

we can compute the direction derivative as

$$\mathrm{D}_{\hat{\mathbf{v}}}(1,1,1) = \nabla h(1,1,1) \cdot \hat{\mathbf{v}} = \frac{1}{21} + \frac{4}{7} + \frac{3}{14} = \frac{23}{42}$$

14.7 Maximum and Minimum Values

18. Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

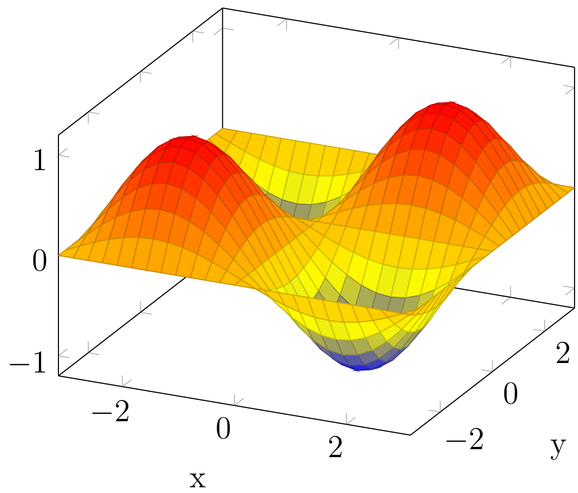
$$f(x,y) = \sin x \sin y, \qquad -\pi < x < \pi, \qquad -\pi < y < \pi$$

$$\implies \begin{cases} f_x = \cos x \sin y \\ f_y = \sin x \cos y \end{cases}$$

$$\implies \begin{cases} f_{xx} = f_{yy} = -\sin x \sin y \\ f_{xy} = f_{yx} = \sin x \sin y \end{cases}$$

$$\implies D = f_{xx} f_{yy} - f_{xy}^2 = 0$$

For $f_x = f_y = 0$, either $x = y = 0$ or $x, y \in \{\pm\pi/2\}$. D does not indicate if f has local extreme values at these critical points.



It is clear that f has 2 local maximums of 1 at $x = y = \pm\pi$ and 2 local minimum of -1 at $x = -y = \pm\pi$, since these are its absolute extreme values as well.

Suppose $f(0,0)$ is a local minimum. Then, by definition, $f(a,b) \geq f(0,0) = 0$ if (a,b) is sufficiently close to origin (say, at most within $[-\pi/2, \pi/2]^2$). However, for all a, b satisfying $ab < 0$, $f(a,b) = \sin a \sin b < 0$, thus our assumption is incorrect. Similarly, f does not has a local maximum at origin because

$$\forall a, b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] : ab > 0, \qquad f(a,b) = \sin a \sin b > 0 = f(0,0)$$

Therefore (0, 0) is a saddle point.

35. Find the absolute extreme values of $f(x,y) = 2x^3 + y^4$ on the unit disc.

The critical points of f occur when

$$f_x = f_y = 0 \iff 6x^2 = 4y^3 = 0 \iff x = y = 0$$

at which $f(x,y) = f(0,0) = 0$.

On the unit circle, as $y^2 = 1 - x^2$, let

$$g(x) = f(x,y) = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1$$

Within $[-1,1]$, $g'(x) = 4x^3 + 6x^2 - 4x = 0$ if and only if $x = 0$ or $x = 0.5$. Since $g(-1) = -2$, $g(0) = 1$, $g(0.5) = 0.8125$ and $g(1) = 2$, the absolute minimum and maximum of g on $[-1,1]$ are respectively $g(-1) = -2$ and $g(1) = 2$.

Thus on the boundary, the minimum value of f is -2 at $(-1, \pm 1)$ and the maximum value is 2 at $(1, \pm 1)$.

46. Find the dimensions of the box with volume 1000 cm^3 that has minimal surface area.

Let the dimensions of the box be x, y, z in dm, x, y, z are positive and $xyz = 1$. Total surface area of the box would then be

$$S(x, y, z) = 2xy + 2yz + 2zx$$

By AM-GM inequality,

$$S(x, y, z) \geq 2 \cdot 3\sqrt{xy \cdot yz \cdot zx} = 6$$

Thus S has its absolute minumum of 6 at $x = y = z = 1$.

53. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

Let the dimensions of the box be three positive numbers x, y, z, $x^2 + y^2 + z^2 = L^2$. The volume of the box would then be $V(x, y, z) = xyz$. By AM-GM inequality,

$$V(x, y, z) = \sqrt{x^2y^2z^2} \leq \frac{x^2 + y^2 + z^2}{3} = \frac{L^2}{3}$$

Thus V has its absolute maximum of $L^2/3$ at $x = y = z = L/\sqrt{3}$.

14.8 Lagrange Multipliers

12. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y, z) = x^4 + y^4 + z^4$ subject to $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

Extreme values of f occur when

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y, z) = 1 \end{cases} \iff \begin{cases} \langle 4x^3, 4y^3, 4z^3 \rangle = \lambda \langle 2x, 2y, 2z \rangle \neq \mathbf{0} \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

1. For $\lambda = 2/3$, $x^2 = y^2 = z^2 = 1/3 = f(x, y, z)$.
2. For $\lambda = 1$ and $(x^2, y^2, z^2) \in \{(0, 1/2, 1/2), (1/2, 0, 1/2), (1/2, 1/2, 0)\}$, $f(x, y, z) = 1/2$.
3. For $\lambda = 2$ and $(x^2, y^2, z^2) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $f(x, y, z) = 1$.

Therefore, subject to the given constrain, f has absolute maximum of 1 and minimum of 1/3.

42. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm.

Let the dimensions of the box be x, y, z in dm, with x, y, z are positive, $2xy + 2yz + 2zx = 15$ and $4x + 4y + 4z = 20$. From these constrains, we can easily obtain $x + y = 5 - z$ and

$$xy + (x + y)z = \frac{15}{2} \iff xy = \frac{15}{2} - 5z + z^2$$

Thus with $0 < z < 5$ the volume of the box is

$$V = xyz = z^3 - 5z^2 + \frac{15z}{2}$$

whose critical points are

$$\frac{\mathrm{d}V}{\mathrm{d}z} = 3z^2 - 10z + \frac{15}{2} = 0 \iff z = \frac{10 \pm \sqrt{10}}{6}$$

at which $V = \dfrac{175 \pm 5\sqrt{10}}{54}$.

On the other hand, the constrains are equivalent to

$$\begin{cases} x^2 + y^2 + z^2 = 10 \\ x + y + z = 5 \end{cases}$$

or the intersection of a sphere and a plane, which result in a circle C. Hence the range of z would be between a and b, whereas each of $z = a$ and $z = b$ only has one point in common with C. Since all surfaces $x^2 + y^2 + z^2 = 10$, $x + y + z = 5$, $z = a$ and $z = b$ has $x = y$ as their plane of symmetry, these two points must be on $x = y$ as well:

$$\begin{aligned} \begin{cases} 2x^2 + z^2 = 10 \\ 2x + z = 5 \end{cases} &\iff \begin{cases} 2x^2 + (5 - 2x)^2 = 10 \\ z = 5 - 2x \end{cases} \\ &\iff \begin{cases} 6x^2 - 20x + 15 = 0 \\ z = 5 - 2x \end{cases} \\ &\iff \begin{cases} x = \dfrac{10 \pm \sqrt{10}}{6} \\ z = \dfrac{5 \pm \sqrt{10}}{3} \end{cases} \\ &\implies V = \frac{175 \pm 5\sqrt{10}}{54} \end{aligned}$$

These are the maximum and minimum volumes of the given box.

15 Multiple Integrals

15.2 Interated Integrals

19. Calculate the double integral.

$$\begin{aligned}
\int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{\pi/6} \left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3}\,\mathrm{d}x \\
&= \int_0^{\pi/6} x\left(\cos x - \cos\left(x+\frac{\pi}{3}\right)\right)\mathrm{d}x \\
&= \int_0^{\pi/6} x\cos\left(x-\frac{\pi}{3}\right)\mathrm{d}x \\
&= \int_0^{\pi/6} x\,\mathrm{d}\cos\left(x-\frac{\pi}{3}\right) \\
&= \left[x\sin\left(x-\frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6}\sin\left(x-\frac{\pi}{3}\right)\mathrm{d}x \\
&= -\frac{\pi}{12} + \left[\cos\left(x-\frac{\pi}{3}\right)\right]_0^{\pi/6} \\
&= \frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{\pi}{12}
\end{aligned}$$

28. Find the volume of the solid enclosed by the surface $z = 1 + e^x \sin y$ and the planes $x = \pm 1$, $y = 0$, $y = \pi$ and $z = 0$.

$$\begin{aligned}
\int_0^{\pi}\int_{-1}^{1}(1+e^x\sin y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\pi}\left[x+e^x\sin y\right]_{x=-1}^{x=1}\,\mathrm{d}y \\
&= \int_0^{\pi}\left(2+\left(e-\frac{1}{e}\right)\sin y\right)\mathrm{d}y \\
&= \left[2x+\left(\frac{1}{e}-e\right)\cos y\right]_0^{\pi} \\
&= 2\pi
\end{aligned}$$

15.3 Double Integrals over General Regions

10. Evaluate the double integral.

$$\begin{aligned}
\int_1^e \int_0^{\ln x} x^3 \, \mathrm{d}y \, \mathrm{d}x &= \int_1^e x^3 \ln x \, \mathrm{d}x \\
&= \int_1^e \ln x \, \mathrm{d}\frac{x^4}{4} \\
&= \left. \frac{x^4 \ln x}{4} \right]_1^e - \int_1^e \frac{x^4}{4} \, \mathrm{d}\ln x \\
&= e^4 - \int_1^e \frac{x^3}{4} \, \mathrm{d}x \\
&= e^4 - \left. \frac{x^4}{16} \right]_1^e \\
&= \frac{15e^4 + 1}{16}
\end{aligned}$$

16. Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$$I = \iint_D y^2 e^{xy} \, \mathrm{d}A, \qquad D \text{ is bounded by } y = x, y = 4, x = 0$$

$$\implies I = \int_0^4 \int_x^4 y^2 e^{xy} \, \mathrm{d}y \, \mathrm{d}x = \int_0^4 \int_0^y y^2 e^{xy} \, \mathrm{d}x \, \mathrm{d}y$$

Since $y^2 e^{xy}$ is simply an exponential function of x, it would be easier to evaluate

$$\begin{aligned}
I &= \int_0^4 \int_0^y y^2 e^{xy} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^4 \left[y^3 e^{xy} \right]_{x=0}^{x=y} \mathrm{d}y \\
&= \int_0^4 y^3 e^{y^2} \, \mathrm{d}y = \int_0^4 y^2 \, \mathrm{d}\frac{e^{y^2}}{2} \\
&= \left. \frac{y^2 e^{y^2}}{2} \right]_0^4 - \int_0^4 \frac{e^{y^2}}{2} \, \mathrm{d}y^2 \\
&= 8e^{16} - \int_0^{16} \frac{e^z}{2} \, \mathrm{d}z \\
&= 8e^{16} - \left. \frac{e^z}{2} \right]_0^{16} = \frac{15e^{16}}{2}
\end{aligned}$$

31. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the plane $y = z$ in the first octant.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y \,\mathrm{d}y \,\mathrm{d}x = \int_0^1 \frac{1-x^2}{2} \,\mathrm{d}x = \frac{1}{3}$$

15.4 Double Integrals in Polar Coordinates

13. Evaluate the given integral by changing to polar coordinates.

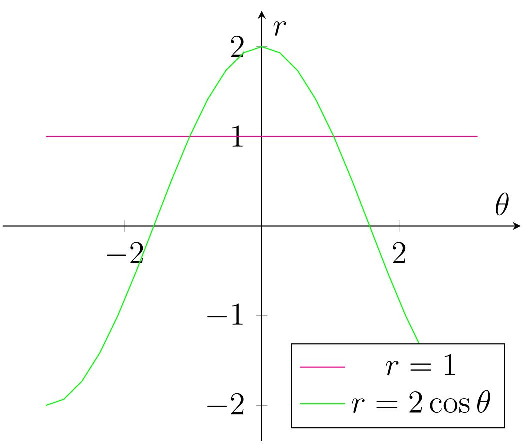
$$I = \iint_R \arctan \frac{y}{x} \,\mathrm{d}A, \qquad \text{where } R = \{(x, y) \,|\, 1 \le x^2 + y^2 \le 4, 0 \le y \le x\}$$

In polar coordinates,

$$R = [1, 2] \times \left[0, \frac{\pi}{4}\right]$$

thus

$$\begin{aligned} I &= \int_0^{\pi/4} \int_1^2 \arctan \frac{r \sin\theta}{r\cos\theta} r \,\mathrm{d}r \,\mathrm{d}\theta \\ &= \int_0^{\pi/4} \int_1^2 \arctan \tan\theta r \,\mathrm{d}r \,\mathrm{d}\theta \\ &= \int_0^{\pi/4} \int_1^2 \theta r \,\mathrm{d}r \,\mathrm{d}\theta \\ &= \int_0^{\pi/4} \frac{3\theta}{2} \,\mathrm{d}r \,\mathrm{d}\theta \\ &= \frac{3\pi^2}{64} \end{aligned}$$



17. Use a double integral to find the area of the region inside $C_1 : (x-1)^2 + y^2 = 1$ and outside $C_0 : x^2 + y^2 = 1$.

In polar coordinates C_1 has the equation $r = 2\cos\theta$ and the equation of C_0 is $r = 1$. Therefore the given region is within $1 \le r \le 2\cos\theta$, whereas $\theta \in [-\pi, \pi]$.

Since on $[-\pi, \pi]$, $2\cos\theta \geq 1 \iff -\pi/3 \leq \theta \leq \pi/3$, the area of the given region is

$$\begin{aligned}
\int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta &= \int_{-\pi/3}^{\pi/3} \frac{4\cos^2\theta - 1}{2}\,\mathrm{d}\theta \\
&= \int_{-\pi/3}^{\pi/3} \left(2\cos^2\theta - 1 + \frac{1}{2}\right)\mathrm{d}\theta \\
&= \int_{-\pi/3}^{\pi/3} \left(\cos 2\theta + \frac{1}{2}\right)\mathrm{d}\theta \\
&= \left[\frac{\sin 2\theta + \theta}{2}\right]_{-\pi/3}^{\pi/3} \\
&= \frac{\sqrt{3}}{2} + \frac{\pi}{3}
\end{aligned}$$

15.5 Applications of Double Integrals

5. Find the mass and center of mass of the lamina that occupies the region triangular D with vertices (0, 0), (2, 1), (0, 3) and has the given density function $\rho(x, y) = x + y$.

$$\begin{aligned}
m &= \iint_D (x+y)\,\mathrm{d}A \\
&= \int_0^2 \int_{x/2}^{3-x} (x+y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 \frac{36 - 9x^2}{8}\,\mathrm{d}x \\
&= 9 - 3 = 6
\end{aligned}$$

$$\begin{aligned}
\bar{x} &= \iint_D \frac{x(x+y)}{m}\,\mathrm{d}A \\
&= \int_0^2 \int_{x/2}^{3-x} \frac{x^2 + xy}{6}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 \frac{12x - 3x^3}{16}\,\mathrm{d}x \\
&= \frac{3}{4}
\end{aligned}$$

$$\begin{aligned}
\bar{y} &= \iint_D \frac{y(x+y)}{m}\,\mathrm{d}A \\
&= \int_0^2 \int_{x/2}^{3-x} \frac{xy + y^2}{6}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 \frac{6 - 3x}{4}\,\mathrm{d}x \\
&= \frac{3}{2}
\end{aligned}$$

15.6 Surface Area

7. Find the area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$\begin{aligned}
&\iint_D \sqrt{1 + \left(\frac{\partial(y^2 - x^2)}{\partial x}\right)^2 + \left(\frac{\partial(y^2 - x^2)}{\partial y}\right)^2}\, \mathrm{d}A \\
=&\iint_D \sqrt{1 + 4x^2 + 4y^2}\, \mathrm{d}A \\
=&\int_0^{2\pi} \int_1^2 r\sqrt{1 + 4r^2\cos^2\theta + 4r^2\sin^2\theta}\, \mathrm{d}r\, \mathrm{d}\theta \\
=&\int_1^2 \pi\sqrt{1 + 4r^2}\, \mathrm{d}r^2 \\
=&\int_1^4 \pi\sqrt{1 + 4t}\, \mathrm{d}t \\
=&\pi\left[\frac{(1 + 4t)^{1.5}}{6}\right]_1^4 \\
=&\frac{17^{1.5} - 5^{1.5}}{6}\pi
\end{aligned}$$

16 Vector Calculus

16.2 Line Integrals

12. Evaluate the integral, where C is the given curve.

$$I = \int_C (x^2 + y^2 + z^2)\, \mathrm{d}s, \qquad C : x = t, y = \cos 2t, z = \sin 2t, 0 \le t \le 2\pi$$

$$\begin{aligned}
I &= \int_0^{2\pi} (x^2 + y^2 + z^2)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, \mathrm{d}t \\
&= \int_0^{2\pi} (t^2 + \cos^2 2t + \sin^2 2t)\sqrt{\left(\frac{\mathrm{d}t}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\cos 2t}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\sin 2t}{\mathrm{d}t}\right)^2}\, \mathrm{d}t \\
&= \int_0^{2\pi} (t^2 + 1)\sqrt{2}\, \mathrm{d}t = \frac{8\pi\sqrt{2}}{3} + 2\pi\sqrt{2}
\end{aligned}$$

15. With C is the line segment from (1, 0, 0) to (4, 1, 2), $x = 3t + 1$, $y = t$, $z = 2t$, whereas $0 \le t \le 1$ and

$$\begin{aligned}
J &= \int_C z^2 \,\mathrm{d}x + x^2 \,\mathrm{d}y + y^2 \,\mathrm{d}z \\
&= \int_0^1 z^2 \frac{\mathrm{d}x}{\mathrm{d}t} \,\mathrm{d}t + x^2 \frac{\mathrm{d}y}{\mathrm{d}t} \,\mathrm{d}t + y^2 \frac{\mathrm{d}z}{\mathrm{d}t} \,\mathrm{d}t \\
&= \int_0^1 (x^2 + 2y^2 + 3z^2) \,\mathrm{d}t \\
&= \int_0^1 (9t^2 + 6t + 1 + 2t^2 + 12t^2) \,\mathrm{d}t \\
&= \int_0^1 (23t^2 + 6t + 1) \,\mathrm{d}t \\
&= \left[\frac{23t^3}{3} + 3t^2 + t \right]_0^1 = \frac{35}{3}
\end{aligned}$$

39. Find the work done by the force field $\mathbf{F}(x, y) = \langle x, y + 2 \rangle$ is moving an object along an arch of the cycloid $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$, $0 \le t \le 2\pi$.

$$\begin{aligned}
W &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \,\mathrm{d}t \\
&= \int_0^{2\pi} \langle x, y + 2 \rangle \cdot \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t} \right\rangle \,\mathrm{d}t \\
&= \int_0^{2\pi} \langle t - \sin t, 3 - \cos t \rangle \cdot \langle 1 - \cos t, \sin t \rangle \,\mathrm{d}t \\
&= \int_0^{2\pi} (t - t\cos t + 2\sin t) \,\mathrm{d}t \\
&= \left[\frac{t^2}{2} - t\sin t - 3\cos t \right]_0^{2\pi} = 2\pi^2
\end{aligned}$$

16.3 The Fundamental Theorem for Line Integral

19. Show that the line integral is independent from any path C from (1, 0) to (2, 1) and evaluate the integral.

$$\int_C \frac{2x}{e^y} \,\mathrm{d}x + \left(2y - \frac{x^2}{e^y} \right) \,\mathrm{d}y = \int_C \left(\frac{2x}{e^y} \hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - \frac{x^2}{e^y} \hat{\mathbf{j}} \right) \cdot \mathrm{d}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}})$$

Since on $\mathbb{R}^2$

$$\frac{\partial}{\partial y}\frac{2x}{e^y} = \frac{-2x}{e^y} = \frac{\partial}{\partial x}\left(2y - \frac{x^2}{e^y}\right)$$

the function

$$\mathbf{F}(x, y) = \frac{2x}{e^y}\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - \frac{x^2}{e^y}\hat{\mathbf{j}}$$

is conservative and thus the given line integral is independent from path.

Let f be a differentiable of (x, y) that $\nabla f = \mathbf{F}$. One function satisfying this is

$$f(x, y) = y^2 + \frac{x^2}{e^y}$$

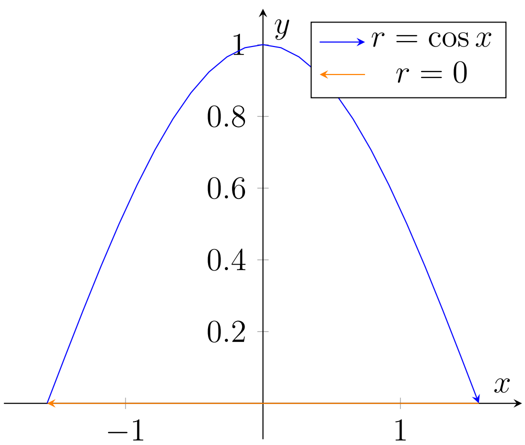
By the fundamental theorem for line integrals,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(2, 1) - f(1, 0) = \frac{4}{e}$$

16.4 Green's Theorem

6. Use Green's Theorem to evaluate the line integral along the given positively oriented rectangle with vertices (0, 0), (5, 0), (5, 2) and (0, 2).

$$\begin{aligned}\int_C \cos y\,\mathrm{d}x + x^2\sin y\,\mathrm{d}y &= \int_0^5\int_0^2\left(\frac{\partial x^2\sin y}{\partial x} - \frac{\partial\cos y}{\partial y}\right)\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^5\int_0^2(2x\sin y + \sin y)\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^5(2x+1)(1-\cos 2)\,\mathrm{d}x\\ &= 30 - 30\cos 2\end{aligned}$$



12. Use Green's Theorem to evaluate the line integral along the path C including the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment connecting these two points.

Since the curve is negatively oriented, by Green's Theorem,

$$
\begin{aligned}
&\int_C (e^{-x} + y^2)\,\mathrm{d}x + (e^{-y} + x^2)\,\mathrm{d}y \\
&= -\int_{-\pi/2}^{\pi/2}\int_0^{\cos x} \left(\frac{\partial}{\partial x}(e^{-y} + x^2) - \frac{\partial}{\partial y}(e^{-x} + y^2)\right)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\pi/2}^{-\pi/2}\int_0^{\cos x} (2x - 2y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\pi/2}^{-\pi/2} (2x\cos x - \cos^2 x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 2x + 1)\,\mathrm{d}x - \int_{-\pi/2}^{\pi/2} 2x\,\mathrm{d}\sin x \\
&= \left[\frac{\sin 2x}{4} + \frac{x}{2} - 2x\sin x - 2\cos x\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2}
\end{aligned}
$$

16.5 Curl and Divergence

This section is to aid my revision of Electromagnetism. First, on $\mathbb{R}^3$, we define

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

then the curl of vector field $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ is

$$
\begin{aligned}
\mathrm{curl}\mathbf{F} = \nabla \times \mathbf{F} = &\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \hat{\mathbf{i}}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)
\end{aligned}
$$

If f is a function of three variables that has continuous second-order partial derivatives, then $\mathrm{curl}(\nabla f) = \mathbf{0}$.

On the other hand, if $\mathrm{curl}\mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is a conservative vector field (preconditions: P, Q and R must be partially differentiable).

Similarly, the divergence of vector field $\mathbf{F}$ is defined as

$$\mathrm{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \hat{\mathbf{i}}\frac{\partial P}{\partial x} + \hat{\mathbf{j}}\frac{\partial Q}{\partial y} + \hat{\mathbf{k}}\frac{\partial R}{\partial z}$$

Trivially, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ because the terms cancel in pairs by Clairaut's Theorem.

The cool thing about operators is that they can be weirdly combined, e.g. $\operatorname{div}(\nabla f) = \nabla \cdot \nabla f = \nabla^2 f$ and $\nabla^2 F = \nabla \cdot \nabla \cdot \mathbf{F}$.

Now we are able to write Green's Theorem in the vector form

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\operatorname{curl}\mathbf{F}) \cdot \hat{\mathbf{k}} \, \mathrm{d}A$$

whereas $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$. The outward normal vector to the contour is given by $\mathbf{n}(t) = \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\mathbf{i}} - \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{j}}$. So we have the second vector form of Green's Theorem.

$$\oint_{\partial S} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}s = \iint_S \operatorname{div}\mathbf{F} \, \mathrm{d}A$$

16.6 Parametric Surfaces and Their Areas

42. Find the area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$.

$$\begin{aligned}
&\int_0^1 \int_{x^2}^{x} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)_2} \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \int_{x^2}^{x} \sqrt{2} \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 (x - x^2)\sqrt{2} \, \mathrm{d}y \, \mathrm{d}x \\
&= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
\end{aligned}$$

17 Second-Order Differential Equations

17.1 Homogeneous Linear Equations

11. Solve the differential equation.

$$2\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0$$

Since the auxiliary equation $2r^2 + 2x - 1 = 0$ has two real and distinct roots $\frac{\pm\sqrt{3} - 1}{2}$, the general solution is

$$y = c_1 \exp \frac{\sqrt{3} - 1}{2} t + c_2 \exp \frac{-\sqrt{3} - 1}{2} t$$

21. Solve the initial value problem.

$$y'' - 6y' + 10y = 0, \qquad y(0) = 2, \qquad y''(0) = 3$$

Since the auxiliary equation $r^2 - 6x + 10 = 0$ has two complex roots $3 \pm i$, the general solution is

$$y = e^{3x}(c_1 \cos x + c_2 \sin x) \Longrightarrow y' = e^{3x}((3c_1 + c_2)\cos x + (3c_2 - c_1)\sin x)$$

As $y(0) = 2$, $c_1 = 2$. Similarly, from $y'(0) = 3$, we can obtain $3c_1 - c_2 = 3 \Longrightarrow c_2 = 3$. Thus the solution of the initial value problem is $y = e^{3x}(3\cos x + 2\sin x)$.

17.2 Nonhomogeneous Linear Equations

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$y'' - 4y' + 5y = e^{-x} \tag{5}$$

The auxiliary equation of $y'' - 4y' + 5y = 0$ is $r^2 - 4r + 5 = 0$ with roots $r = 2 \pm i$. Hence the solution to the complementary equation is

$$y_c = e^{2x}(c_1 \cos x + c_2 \sin x)$$

Since $G(x) = e^{-x}$ is an exponential function, we seek a particular solution of an exponential function as well:

$$y_p = Ae^{-x} \Longrightarrow y_p' = -Ae^{-x} \Longrightarrow y_p'' = Ae^{-x}$$

Substituting these into the differential equation, we get

$$Ae^{-x} - 4Ae^{-x} + 5Ae^{-x} = e^{-x} \iff A = \frac{1}{10}$$

Thus the general solution of the exponential equation is

$$y = y_c + y_p = e^{2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{10e^x}$$

$$y'' + y' - 2y = x + \sin 2x, \qquad y(0) = 1, \qquad y'(0) = 0 \tag{10}$$

The auxiliary equation of $y'' + y' - 2y = 0$ is $r^2 + r - 2 = 0$ with roots $r = -2, 1$. Thus the solution to the complementary equation is

$$y_c = c_1 e^x + \frac{c_2}{e^{2x}}$$

We seek a particular solution of the form

$$y_p = Ax + B + C\cos 2x + D\sin 2x \implies y_p' = A - 2C\sin 2x + 2D\cos 2x \implies y_p'' = -4C\cos 2x - 4D\sin 2x$$

Substituting these into the differential equation, we get

$$(-4C+2D-2C)\cos 2x + (-4D-2C-2D)\sin 2x + A - 2B - 2Ax = x + \sin 2x$$
$$\iff \begin{cases} -4C + 2D - 2C = 0 \\ -4D - 2C - 2D = 1 \\ A - 2B = 0 \\ -2A = 1 \end{cases} \iff \begin{cases} A = -1/2 \\ B = -1/4 \\ C = -1/20 \\ D = -3/20 \end{cases}$$

Thus the general solution of the exponential equation is

$$y = y_c + y_p = c_1e^x + \frac{c_2}{e^{2x}} - \frac{x}{2} - \frac{1}{4} - \frac{\cos 2x}{20} - \frac{3\sin 2x}{20} \implies y' = c_1e^x - \frac{2c_2}{e^{2x}} - \frac{1}{2} + \frac{\sin 2x}{10} - \frac{3\cos 2x}{10}$$

Since $y(0) = 1$ and $y'(0) = 0$,

$$\begin{cases} c_1 + c_2 - \frac{1}{4} - \frac{1}{20} = 1 \\ c_1 - 2c_2 - \frac{3}{10} = 0 \end{cases} \iff \begin{cases} c_1 + c_2 = \frac{13}{10} \\ c_1 - 2c_2 = \frac{3}{10} \end{cases} \iff \begin{cases} c_1 = \frac{29}{30} \\ c_2 = \frac{1}{3} \end{cases}$$

Therefore the solution to the initial value problem is

$$y = \frac{29e^x}{30} + \frac{1}{3e^{2x}} - \frac{x}{2} - \frac{1}{4} - \frac{\cos 2x}{20} - \frac{3\sin 2x}{20}$$

17.3 Applications

3. A spring with a mass of 2 kg has damping constant 14, and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is stretched 1 m beyond its natural length and then released with zero velocity. Find the position of the mass at any time t.

By Hooke's law,

$$k(0.5) = 6 \iff k = 12$$

By Newton's second law of motion,

$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 14\frac{\mathrm{d}x}{\mathrm{d}t} + 12x = 0$$

Since the auxiliary equation $2r^2 + 14r + 12 = 0$ has two real and distinct roots $r = -6, -1$, the general solution is

$$x = \frac{c_1}{e^t} + \frac{c_2}{e^{6t}} \Longrightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-c_1}{e^t} - \frac{6c_2}{e^{6t}}$$

From $x(0) = 1$ and $x'(0) = 0$ we get

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 - 6c_2 = 0 \end{cases} \iff \begin{cases} c_1 = 6/5 \\ c_2 = -1/5 \end{cases}$$

Therefore the position at any time t is

$$x = \frac{6}{5e^t} - \frac{c_2}{5e^{6t}}$$

9 First-Order Differential Equations

9.3 Separable Equations

8. Solve the differential equation.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{e^y \sin^2\theta}{y\sec\theta} \\ \iff \int \frac{y}{e^y}\,\mathrm{d}y &= \int \sin\theta\cos\theta\,\mathrm{d}\theta \\ \iff \int -y\,\mathrm{d}e^{-y} &= \int \sin^2\theta\,\mathrm{d}\sin\theta \\ \iff \int e^{-y}\,\mathrm{d}y - \frac{y}{e^y} &= \frac{\sin^3\theta}{3} \\ \iff \frac{1+y}{e^y} &= C - \frac{\sin^3\theta}{3} \end{aligned}$$

9.5 Linear Equations

28. In a damped RL circuit, the generator supplies a voltage of $E(t) = 40\sin 60t$ volts, the inductance is 1 H, the resistance is 10 Ω and $I(0) = 1$ A.

$$\begin{aligned} & E - L\frac{\mathrm{d}I}{\mathrm{d}t} - RI = 0 \\ \iff & \frac{40}{L}\sin 60t = \frac{\mathrm{d}I}{\mathrm{d}t} + \frac{RI}{L} \\ \iff & \frac{40e^{tR/L}}{L}\sin 60t = \frac{RI}{L}e^{tR/L} + \frac{\mathrm{d}I}{\mathrm{d}t}e^{tR/L} \\ \iff & \frac{40}{L}\int e^{tR/L}\sin 60t\,\mathrm{d}t = \int \mathrm{d}Ie^{tR/L} \end{aligned} \qquad (*)$$

Let $J = \int e^{tR/L}\sin 60t\,\mathrm{d}t$,

$$\begin{aligned} J &= \frac{-1}{60}\int e^{tR/L}\,\mathrm{d}\cos 60t \\ &= \frac{1}{60}\int \cos 60t\,\mathrm{d}e^{tR/L} - \frac{e^{tR/L}\cos 60t}{60} \\ &= \frac{R}{3600L}\int e^{tR/L}\,\mathrm{d}\sin 60t - \frac{e^{tR/L}\cos 60t}{60} \\ &= \frac{R}{3600L}e^{tR/L}\sin 60t - \frac{R}{3600L}\int \sin 60t\,\mathrm{d}e^{tR/L} - \frac{e^{tR/L}\cos 60t}{60} \\ &= \frac{R}{3600L}e^{tR/L}\sin 60t - \frac{R^2}{3600L^2}J - \frac{e^{tR/L}\cos 60t}{60} \end{aligned}$$

Hence $J = \dfrac{e^{tR/L}(RL\sin 60t - 60L^2\cos 60t)}{R^2 + 3600L^2}$ and $(*)$ is equivalent to

$$\begin{aligned} \frac{40e^{tR/L}(R\sin 60t - 60L\cos 60t)}{R^2 + 3600L^2} &= Ie^{tR/L} - C \\ \iff I &= \frac{40R\sin 60t - 2400L\cos 60t}{R^2 + 3600L^2} + \frac{C}{e^{tR/L}} \\ \iff I &= \frac{\sin 60t - 3\cos 60t}{5} + \frac{C}{e^{t/20}} \end{aligned}$$

Since $I = 1$ at $t = 0$,

$$1 = \frac{\sin 0 - 3\cos 0}{5} + \frac{C}{e^0} \iff C = \frac{8}{5}$$

and thus $I = \dfrac{\sin 60t - 3\cos 60t}{5} + \dfrac{8}{5}\exp\dfrac{-t}{20}$.

At $t = 0.1$, $I = (\sin 6 - 3\cos 6)/5 + 1.6e^{-1/200} \approx 2.11$ A.