CHAPTER I. Lesson No.1. Basic Probability 1

1. (a) Consider rolling a six-sided die. Let A be the set of outcomes where the roll is an even number. Let B be the set of outcomes where the roll is greater than 3. Calculate and compare the sets on both sides of De Morgan's law.
    - $(A \cup B)^c = A^c \cap B^c$
    - $(A \cap B)^c = A^c \cup B^c$

   (b) Prove that: $P(A^c \cap B^c) = 1 - P(A) - P(B) + P(A \cap B)$

   (c) Consider events A and B such that $P(A) = 1/2, P(A \cup B) = 3/4, P(B^c) = 5/8$. Find $P(A \cap B)$, $P(A^c \cap B^c)$, $P(A^c \cup B^c)$ and $P(B \cap A^c)$?

2. A four-sided die is rolled repeatedly, until the first time (if ever) that an even number is obtained. What is the sample space for this experiment?

3. A ball is drawn at random from a box containing 6 red balls, 4 white balls, and 5 blue balls. Determine the probability that it is

   (a) *not* red

   (b) red *or* white

4. A student is going to graduate from an industrial engineering department in a university by the end of the semeter. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from the first company C_a is 0.8, and the probability that he gets an offer from the second one C_b is 0.6. If, on the hand, he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

5. Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover?

6. Three friends, Rick, Brenda and Ali, go to a football match but forget to say which entrance to the ground they will meet at. There are four entrances, A, B,C and D. Each friend chooses an entrance independently.

- The probability that Rick chooses entrance A is $\frac{1}{3}$. The probabilities that he chooses entrances B, C or D are all equal.
- Brenda is equally likely to choose any of the four entrances.
- The probability that Ali chooses entrance C is $\frac{2}{7}$ and the probability that he chooses entrance D is $\frac{3}{5}$. The probabilities that he chooses the other two entrances are equal.

(a) Find the probability that at least 2 friends will choose entrance B.

(b) Find the probability that the three friends will all choose the same entrance.

7. We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

   (a) Find the probability that doubles are rolled?

   (b) Given that the roll results in a sum of 4 or less, find the conditional probability that doubles are rolled?

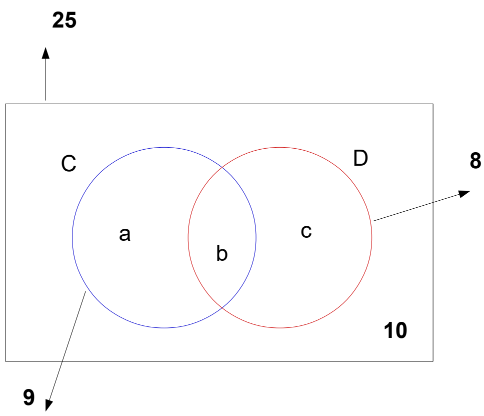
   (c) Find the probability that at least one die roll is a 6?

8. A baby rolls two dice. It's assumed that there are two fair 6-sided dice.

   (a) Find the probability of getting a sum of 7?

   (b) Then, calculate the probability of not getting a sum of 7 or 11?

9. In a group of 25 boys, nine are members of the chess club (C), eight are members of the debating club (D) and 10 are members of neither of these clubs. This information



is shown in the Venn diagram.

(a) Find the values of a, b and c.

(b) Find the probability that a randomly selected boy is:

- a member of the chess club or the debating club
- a member of exactly one of these clubs.

10. Each of 27 tourists was asked which of the countries Angola (A), Burundi (B) and Cameroon (C) they had visited. Of the group, 15 had visited Angola, 8 had visited Burundi, 12 had visited Cameroon, 2 had visited all three countries, and 21 had visited only one. Of those who had visited Angola, 4 had visited only one other country. Of those who had not visited Angola, 5 had visited Burundi only. All of the tourists had visited at least one of these countries.

(a) Draw a fully labelled Venn diagram to illustrate this information.

(b) Find the number of tourists in set B' and describe them.

(c) Describe the tourists in set $(A \cap B) \cup C'$ and state how many there are.

(d) Find the probability that a randomly selected tourist from this group had visited at least two of these three countries.

CHAPTER I. Lesson No.2. Basic Probability 2

1. A bin contains 5 defective (that immediately fail when put in use), 10 partially defective (that fail after a couple of hours of use), and 25 acceptable transistors. A transistor is chosen at random from the bin and put into use. If it does not immediately fail, what is the probability it is acceptable?

2. (a) Tossing a coin n times, what is the probability of getting at least one head?
   (b) Tossing a coin 4 times, what is the probability of getting at least one head?
   (c) Rolling a die 4 times, what is the probability of getting at least one 6?
   (d) How many attempts/tries are need to get the probability of getting at least one 6 of 0.99?

3. A woman travels to work by bicycle 70% of the time and by scooter 30% of the time. If she uses her bicycle she is late 3% of the time but if she uses her scooter she is late only 2% of the time.
   (a) Find the probability that the woman is late for work on any particular day.
   (b) Given that the woman expects not to be late on approximately 223 days in a year, find the number of days in a year on which she works.

4. **Using a biased coin to make an unbiased decision.** Minh and Nam want to choose between the Kpop music and US music by tossing a fair coin. Unfortunately, the only available coin is biased (about the bias is known by 40% and 60%). How can they use the biased coin to make a decision so that either option (Kpop music or US music) is equally likely to be chosen?

5. It is required to put 3-math, 2-history and 4-bio books placed on the shelf. Find the ways in cases:
   (a) no restriction?
   (b) all subjects must stay together?
   (c) only bio must stay together?

6. In how many ways can 6 people be seated at a round table if

   (a) they can sit anywhere?

   (b) 2 particular people cannot sit next to each other?

7. We draw the top 7 cards from a well-shufed standard 52-card deck. Find the probability that:

   (a) The 7 cards include exactly 3 aces.

   (b) The 7 cards include exactly 2 kings.

   (c) The probability that the 7 cards include exactly 3 aces or exactly 2 kings or both.

8. Box I contains 3 red and 2 blue marbles while box II contains 2 red and 8 blue marbles. A fair coin is tossed. If the coin turns up heads, a marble is chosen from box I, if it turns up tails, a marble is chosen from box II. Find the probability that

   (a) A red marble is picked?

   (b) A blue marble is picked?

9. **Birthday Problem.** If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than 0.5?

10. **The Prisoner's Dilemma.** The release of two out of three prisoners has been announced. but their identity is kept secret. One of the prisoners considers asking a friendly guard to tell him who is the prisoner other than himself that will be released, but hesitates based on the following rationale: at the prisoner's present state of knowledge, the probability of being released is 2/3, but after he knows the answer, the probability of being released will become 1/2, since there will be two prisoners (including himself) whose fate is unknown and exactly one of the two will be released. What is wrong with this line of reasoning?

CHAPTER II. Lesson No.3. Discrete Random Variable 1

A. Discrete random variables and PMF

1. Let X represent the number of heads that can come up when tossing a coin in two cases: twice and thrice times. For each case, find the PMF $(p_X(x))$ corresponding to the random variable X?

2. Suppose that a pair of fair dice are to be tossed, and let the random variable X denote the sum of the points.

   (a) Obtain the the PMF $(p_X(x))$ corresponding to the random variable X?

   (b) Draw the graph of $p_X(x)$?

3. The MIT soccer team has 2 games scheduled for one weekend. It has a 0.4 probability of not losing the first game, and a 0.7 probability of not losing the second game, independent of the first. If it does not lose a particular game, the team is equally likely to win or tie, independent of what happens in the other game. The MIT team will receive 2 points for a win, 1 for a tie, and 0 for a loss. Find the PMF of the number of points that the team earns over the weekend?

B. Expectation of random variables

4. A random variable X is defined by

$$X = \begin{cases} -2 & \text{prob.}1/3 \\ 3 & \text{prob.}1/2 \\ 1 & \text{prob.}1/6 \end{cases}$$

   Find $E[X]$, $E[2X+5]$ and $E[X^2]$

5. Find the expected value for the number of girls for a family with three children?

6. A game is played using one die. If the die is rolled and shows $1, 2$ or 3, the player wins nothing. If the die shows 4 or 5, the player wins 3 dollars. If the die shows 6,

the player wins 9 dollars. If there is a charge of 1 dollar to play the game, what is the game's expected value? Describe what this means in practical terms?

7. In a lottery there are 200 prizes of 5 dollars, 20 prizes of 25 dollars, and 5 prizes of 100 dollars. Assuming that $10,000$ tickets are to be issued and sold, what is the expected value for a ticket's value?

8. In a raffle, there is a grand prize/first prize of $2,000$ dollars, a second prize is $1,000$ dollars, two third prizes for 500 dollars and five fourth prizes at 100 dollars. If $5,000$ tickets are sold with 2 dollars per one, what is the expected value of a ticket?

C. Variance and standard deviation

9. What is the mean and variance ($var[X]$) when X represents the outcome when we roll a fair die?

10. Find the variance and the standard deviation of the sum obtained in tossing a pair of fair dice?

11. For the random variable X with PMF

$$p_X(x) = \begin{cases} \frac{1}{9} & \text{if } x \text{ is an integer in the range } [-4,4] \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance of X?

12. Let X be a random variable with PMF

$$p_X(x) = \begin{cases} \frac{x^2}{a} & \text{if } x \text{ is an integer in the range } [-3,3] \\ 0 & \text{otherwise} \end{cases}$$

(a) Find a and $E[X]$?

(b) What is the PMF of the random variable $Z = (X - E[X])^2$?

(c) Using the result from part (b), find the variance of X?

CHAPTER II. Lesson No.4. Discrete Random Variable 2

A. PMF, Expectation, and Conditional PMF & Expectation

1. (a) Let X be the roll of a fair six-sided die, and let A be the event that the roll is an number greater than or equal to 4. Find the conditional PMF $p_{X|A}(x)$?

   (b) Let X represent the toss of a coin 3 times, and let B be the event that the toss gets at least two heads. Obtain the conditional PMF $p_{X|B}(x)$?

   (c) Let X be the roll of a pair of fair dice, and let C be the event that the roll denote a sum of 7. Compute PMF $p_{X|C}(x)$?

2. Messages transmitted by a computer in Boston through a data network are destined for New York with probability 0.5, for Chicago with probability 0.3, and for San Francisco with probability 0.2. The transit time X of a message is random. Its mean is 0.05 seconds if it is destined for New York, 0.1 seconds if it is destined for Chicago, and 0.3 seconds if it is destined for San Francisco. Find the conditional expectation $E[X]$?

3. Average Speed Versus Average Time. If the weather is good (which happens with probability 0.6). Alice walks the 2 miles to class at a speed of $v = 5$ miles per hour, and otherwise rides her motorcycle at a speed of $v = 30$ miles per hour.

   (a) What is the expected value of Alice's speed?

   (b) What is the expected value of the time T to get to class?

4. Mean and variance of the geometric. You write a software program over and over, and each time there is probability p that it works correctly, independent of previous attempts. What is the mean and variance of X, the number of tries until the program works correctly?

B. PMF, joint PMF and independent variables

5. Consider two independent coin tosses, each with a 3/4 probability of a head, and let X be the number of heads obtained. Compute the expected value?

6. Alice passes through four traffic lights on her way to work, and each light is equally likely to be green or red, independent of the others. What is the PMF, the mean, and the variance of the number of red lights that Alice encounters?

7. Suppose that n people throw their hats in a box and then each picks one hat at random. Assuming each hat can be picked by only one person, and each asignment of hats to persons is equally likely. What is the expected value of X, the number of people that get back their own hat?

8. Consider four independent rolls of a 6-sided die. Let X be the number of $1s$ and let Y be the number of $2s$ obtained. What is the joint PMF of X and Y?

9. The joint PMF of two discrete random variables X and Y is given by

$$p_{XY}(x,y) = \begin{cases} c(2x+y) & \text{where } \quad 0 \le x \le 2, \quad 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the value of the constant c?

   (b) Find $P(X = 2, Y = 1)$?

   (c) Find $P(X \ge 1, Y \le 2)$?

   (d) Find the marginal PMF of X?

   (e) Find the marginal PMF of Y?

   (f) Show that the random variables X and Y are dependent?

   (g) Find the conditional PMF of y given $X = 2$? Then, compute $P(Y = 1|X = 2)$?

   (h) Find the conditional PMF of x given $Y = 2$? Then, compute $P(X = 3|Y = 2)$?

10. The joint PMF of two discrete random variables X and Y is given by

$$p_{XY}(x,y) = \begin{cases} cxy & \text{where } \quad x = 1,2,3, \quad y = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the value of the constant c?

   (b) Find $P(X = 2, Y = 3)$?

(c) Find $P(1 \leq X \leq 2, Y \leq 2)$?

(d) Find $P(X \geq 2)$, $P(Y < 2)$?, $P(X = 1)$? and $P(Y = 3)$?

(e) Find the marginal PMF of X, Y?

(f) Determine whether X and Y are independent?

CHAPTER II. Lesson No.5. Continuous Random Variable 1

A. Probability Density Function (PDF) and Cumulative Distribution Function (CDF)

1. A probability density function (PDF) such that

$$f_X(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the constant c?

   (b) Compute $P(1 < X < 2)$?

   (c) Find the CDF for the random variable X?

   (d) Use the result of (c) to find $P(1 < X \leq 2)$?

2. Alice's driving time to work is between 15 and 20 minutes if the day is sunny, and between 20 and 25 minutes if the day is rainy, with all time being equally likely in each case. Assume that a day is sunny with probability 2/3 and rainy with probability 1/3. What is the PDF of the driving time, viewed as a random variable X?

3. A random variable X has the PDF $f_X(x) = \dfrac{c}{x^2+1}$, where $-\infty < x < \infty$.

   (a) Find the value of the constant c?

   (b) Find the probability that X^2 lies between 1/3 and 1?

   (c) Find the CDF of X?

4. The CDF for a random variable X is

$$F_X(x) = \begin{cases} 1 - e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

   (a) Find the probability density function?

   (b) Find the probability that $X > 2$?

   (c) Find the probability that $-3 < X \leq 4$?

5. Alice goes to the bank to make a withdrawal, and is equally likely to find 0 or 1 customer ahead of her. The service time of the customer ahead, if present, is exponentially distributed with parameter λ. What is the CDF of Alice's waiting time?

B. Expected value, variance and standard deviation

6. Let X have the PDF,

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}$$

where λ is a positive scalar. Verify that f_X satisfes the normalization condition, and evaluate the mean and variance of X?

7. A continuous random variable X has PDF given by

$$f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

   (a) Find $E[X]$, $E[X^2]$?

   (b) Find the variance and the standard deviation for the random variable of X?

8. The density function of X is given by

$$f_X(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

If $E[X] = 3/5$, find a and b?

9. The lifetime in hours of an electronic tube is a random variable having a probability density function given by $f_X(x) = xe^{-x}$ with $x \geq 0$. Compute the expected value lifetime of such a tube?

10. The lifetime of an automobile battery is described by a random variable X having negative exponential distribution with parameter $\lambda = 1/3$.

    (a) Determine the expected lifetime of the battery, then find the variance and the standard deviation of random variable X?

    (b) Calculate the probability that the lifetime will be between 2 and 4 time units?

Chapter II. Lesson No.6. Continuous Random Variable 2

1. The time (in hours) required to repair a machine is an exponential distributed random variable with paramter $\lambda = 1/2$.

   (a) Find the probability that a repair time exceeds 2 hours?

   (b) Find the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 8 hours?

2. Alice's driving time to work is between 15 and 25 minutes if the day is sunny, and between 20 and 25 minutes if the day is rainy, with all time being equally likely in each case. Assume that a day is sunny with probability 2/3 and rainy with probability 1/3. What is the PDF of the driving time, viewed as a random variable X?

3. The metro train arrives at the station near your home every quarter hour starting at 6.0 a.m. You walk into the station every morning between 7.10 and 7.30 a.m, and your arrival time is a uniform random variable over this interval. What is the PDF of the time you have to wait for the first train to arrive?

4. Let X be a random variable with PDF

$$f_X(x) = \begin{cases} x/4 & 1 < x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

   and let A be the event $(X \geq 2)$

   (a) Find $E[X]$, $P(A)$, $f_{X|A}(x)$ and $E[X|A]$?

   (b) Let $Y = X^2$. Find $E[Y]$ and $var(Y)$?

5. Alice goes to the bank to make a withdrawal, and is equally likely to find 0 or 1 customer ahead of her. The service time of the customer ahead, if present, is exponentially distributed with parameter λ. What is the CDF of Alice's waiting time?

6. The joint PDF of two continuous random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} cxy & 0 < x < 4, \quad 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant c?

(b) Find $P(1 < X < 2, 2 < Y < 3)$?

(c) Find $P(X \geq 3, Y \leq 2)$?

(d) Find the marginal PDFs of X and Y?

(e) Find $P(X + Y < 3)$?

(f) Find the density function of $U = X + 2Y$?

7. The joint PDF of two continuous random variables X and Y is

$$f_{X,Y}(x, y) = \begin{cases} 8xy & 0 \leq x \leq 1, \quad 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal density of X and Y?

(b) Find the conditional density function of X and Y?

(c) Find the conditional expectation of Y given X, X given Y?

(d) Find the conditional variance of Y given X?

8. Let

$$f_{X,Y}(x, y) = \begin{cases} e^{-(x+y)} & x \geq 0, \quad y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

be the joint density function of X and Y. Find the conditional density function of X given Y, Y given X?

Chapter II. Lesson No.7. Advance Topics

A. PDF and CDF

1. If $Z \sim N(0,1)$, find

   (a) $P(Z > 1.2)$

   (b) $P(-2.0 < Z < 2.0)$

   (c) $P(-1.2 < Z < 1.0)$

2. If $X \sim N(4,9)$, find

   (a) $P(X > 6)$

   (b) $P(X > 1)$

3. The annual snowfall at a particular geographic location is modeled as a normal random variable with a mean of $\mu = 60$ inches and a standard deviation of $\sigma = 20$. What is the probability that this year's snowfall will be at least 80 inches?

4. Customers arrive at a garage at an average rate of 2 per five minute period. What is the probability that less than 15 arrive in a one hour period?

B. Covariance and correlation coefficient

5. The joint PMF of two discrete random variables X and Y is given by

$$p_{XY}(x,y) = \begin{cases} c(2x+y) & \text{where} \quad 0 \le x \le 2, \quad 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

   It is noted that x and y can assume all integers. Compute:

   (a) $E[X]$, $E[Y]$, $E[XY]$

   (b) $E[X^2]$, $E[Y^2]$, $var(X)$, $var(Y)$

   (c) $cov(X,Y)$, $\rho(X,Y)$

6. The joint PDF of two continuous random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} c(2x+y) & \text{where} \quad 2 < x < 6, \quad 0 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

Compute:

(a) $E[X]$, $E[Y]$, $E[XY]$

(b) $E[X^2]$, $E[Y^2]$, $var(X)$, $var(Y)$

(c) $cov(X,Y)$, $\rho(X,Y)$

C. Derived distributions

7. Let X be uniform on $[0,1]$, and let $Y = \sqrt{X}$. Find PDF of random variable Y?

8. John is driving from Boston to the New York area, a distance of 180 miles at a constant speed, whose value is uniformly distributed between 30 and 60 miles per hour. What is the PDF of the duration of the trip?

9. The metro train arrives at the station near your home every quarter hour starting at 6.0 a.m. You walk into the station every morning between 7.10 and 7.30 a.m, and your arrival time is a uniform random variable over this interval. Let X be the elapsed time, in minutes, between 7.10 and the time of your arrival. Let Y be the time that you have to wait until you board a train. Calculate the CDF of Y in terms of the CDF of X and differentiate to obtain a formula for the PDF of Y?

Chapter III. Lesson No.9. Limit Theorems

1. Let X be a random variable with $\mu = 10$ and $\sigma = 4$. A sample of size 100 is taken from this population.

   (a) Find the probability that the sample mean of these 100 observations is less than 9?

   (b) Find the probability that the sum of these 100 observations is less than 900?

2. A large freight elevator can transport a maximum of 9800 pounds. Suppose a load of cargo containing 49 boxes must be transported via the elevator. Experience has shown that the weight of boxes of this type of cargo follows a distribution with mean $\mu = 205$ pounds and standard deviation $\sigma = 15$ pounds. Based on this information, what is the probability that all 49 boxes can be safely loaded onto the freight elevator and transported?

3. From past experience, it is known that the number of tickets purchased by a student standing in line at the ticker window for the football match of UCLA against USC follows a distribution that has mean $\mu = 2.4$ and standard deviation $\sigma = 2.0$. Suppose that few hours before the start of one of these matches there are 100 eager students in line to purchase tickets. If only 250 tickets remain, what is the probability that all 100 students will be able to purchase the tickets they desire?

4. The homework consists of 40 independent problems. On the average, it takes 5 minutes to solve a problem, with a standard deviation of 2 minutes. Find the probability that homework will be completed in less than 3 hours?

5. An average scanned image occupies 0.6 megabytes of memory with a standard deviation of 0.4 megabytes. If you plan to install 80 images on your website, what is the probability that their total size is between 47 megabytes and 56 megabytes?

6. The amount of regular unleaded gasoline purchased every week at a gas station near UCLA follows the normal distibution with mean 50000 gallons and standard deviation 10000 gallons. The starting supply of gasoline is 74000 gallons, and there is a scheduled weekly delivery of 47000 gallons ($47000gs$).

(a) Find the probability that, after 11 weeks, the supply of gasoline will be below 20000gs?

(b) How much should the weekly delivery be so that after 11 weeks the probability that the supply is below 20000 gallons is only 0.5%?