Numerical Methods: Heat Transfer

Nguyễn Gia Phong–BI9-184

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Given a bar of length $L = 0.4\,\mathrm{m}$ consisting of homogeneous and isotropic material with the initial temperature of $T_0 = 0\,°\mathrm{C}$. Suppose it is perfectly insulated with the exception of the ends with the temperature of $T_g = 100\,°\mathrm{C}$ and $T_d = 50\,°\mathrm{C}$. The thermal properties of material will be taken constant.

- Specific heat capacity: $c_p = 900\,\dfrac{\mathrm{J}}{\mathrm{kg}\,°\mathrm{C}}$
- Thermal conductivity: $\lambda = 237\,\dfrac{\mathrm{W}}{\mathrm{m}\,°\mathrm{C}}$
- Density: $\rho = 2700\,\dfrac{\mathrm{kg}}{\mathrm{m}^3}$
- Thermal diffusivity: $\alpha = \dfrac{\lambda}{\rho c_p} = 9.753 \times 10^{-5}\,\dfrac{\mathrm{m}^2}{\mathrm{s}}$

The heat transfer in this bar can be described by the following partial differential equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{$*$}$$

where the temperature T is a function of postition x and time t.

To solve the problem numerically, we devide space and time into equal intervals of norms Δx and Δt respectively and let $M = L/\Delta x$. Consequently, the spartial coordinate is defined as $x_i = (i-1)\Delta x$ with $i \in [1\,..\,M+1]$ and the temporal one is $t_n = n\Delta t$ with $n \in \mathbb{N}^*$. With these definitions[1], we denote $T_i^n = T(x_i, t_n)$. Using numerical methods, we may start approximating the solutions of $(*)$.

[1] I believe $i = 0$ and $n = 0$ in the assignment papers are typos since then the domain of x_i would exceed L and t_0 would be negative.

1. The left-hand side of $(*)$ can be approximated as

$$\frac{\partial T}{\partial t} = \frac{T_i^{n+1} - T_i^n}{\Delta t}$$

2. Similarly, the right-hand side is expressed in the following form

$$\alpha \frac{\partial^2 T}{\partial x^2} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$

3. $(*)$ is therefore reformulated as

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$

4. From the formular above and let $\beta = \alpha \frac{\Delta t}{\Delta x^2}$, we get

$$T_i^{n+1} = T_i^n + \beta \left(T_{i+1}^n - 2T_i^n + T_{i-1}^n \right)$$

5. Boundary conditions:
   - $\forall n \in \mathbb{N}^*,\ T_1^n = T(0, t_n) = T_g$
   - $\forall n \in \mathbb{N}^*,\ T_{M+1}^n = T(L, t_n) = T_d$
   - $\forall i \in [2 \mathinner{..} M],\ T_i^1 = T(x_i, 0) = T_0$

6. From (4) and (5), the temperature at point x_i of the bar at time t_n is recursively defined as

$$T_i^n = \begin{cases} T_i^{n-1} + \beta \left(T_{i+1}^{n-1} - 2T_i^{n-1} + T_{i-1}^{n-1} \right) & \text{if } 1 < i \leq M \land n > 1 \\ T_g & \text{if } i = 1 \\ T_d & \text{if } i = M + 1 \\ T_0 & \text{otherwise} \end{cases}$$

   Since the temperature only depends on the values in the past, values within $(i, n) \in [1 \mathinner{..} M+1] \times [1 \mathinner{..} N]$ with any N of choice could be computed via dynamic programming:

   (a) Create a 2-dimensional dynamic array T with one-based index and size $(M + 1) \times 1$

   (b) Initialize T with $T_1^1 = T_g$, $T_{M+1}^1 = T_d$ and $T_i^1 = T_0\ \forall i \in [2 \mathinner{..} M]$, where T_i^n is element of row i and column n

(c) For $n = 2$ to N

- Let $T_1^k = T_g$
- For $i = 2$ to M, let $T_i^k = T_i^{k-1} + \beta\left(T_{i+1}^{k-1} - 2T_i^{k-1} + T_{i-1}^{k-1}\right)$
- Let $T_{M+1}^k = T_d$

(d) Return T_i^n

Each iteration in (c) can be written in matrix notation as $T^k = AT^{k+1}$, where T_n is column n and A is a matrix of size $(M+1) \times (M+1)$

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1-2\beta & \beta & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1-2\beta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-2\beta & \beta & 0 \\ 0 & 0 & 0 & \cdots & \beta & 1-2\beta & \beta \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

7. Steps (a) to (c) is then implemented in Octave as

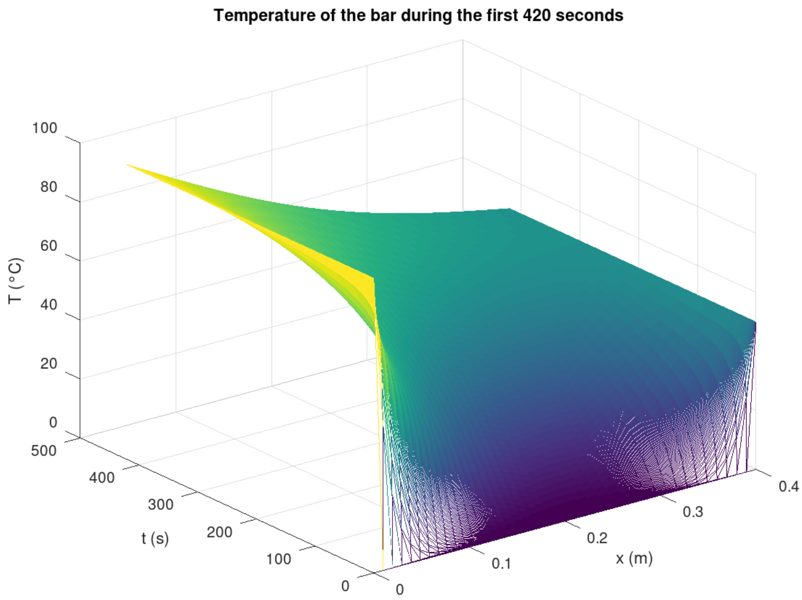
```
function T = heatrans (cp, lambda, rho, Tg, Td, T0, L,
                       dx, dt, N)
  alpha = lambda / rho / cp;
  beta = alpha * dt / dx^2;
  M = round (L / dx);
  side = repelem (beta, M);
  A = (diag (repelem (1 - 2*beta, M + 1))
       + diag (side, -1) + diag (side, 1));
  A(1, :) = A(end, :) = 0;
  A(1, 1) = A(end, end) = 1;

  T = repelem (T0, M + 1);
  [T(1) T(end)] = deal (Tg, Td);
  for k = 2 : N
    T(:, k) = A * T(:, k - 1);
  end
end
```

Choosing $\Delta x = 0.01\,\mathrm{m}$, $\Delta t = 0.5\,\mathrm{s}$ and $N = 841$, we define

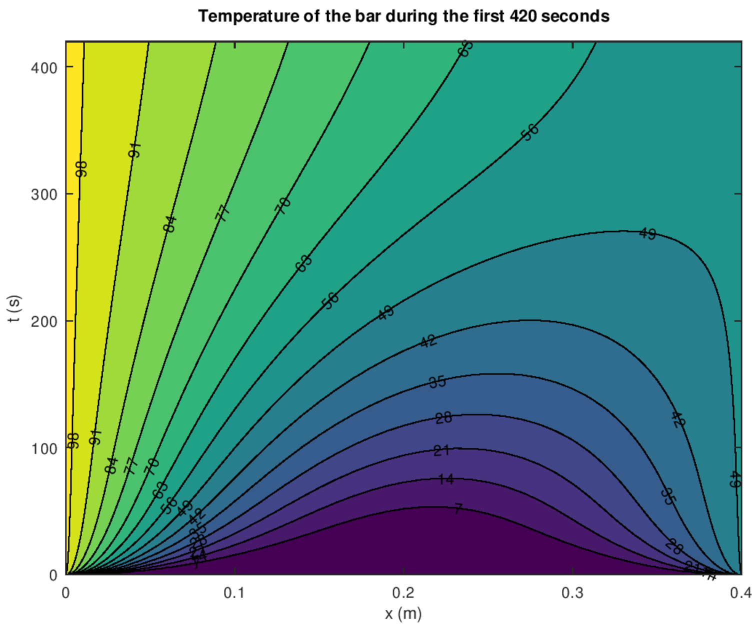
```
T = heatrans (900, 237, 2700, 100, 50, 0, 0.4,
              0.01, 0.5, 841);
```

then the temperature at point x_i at time t_n is `T(i, n)`.

To visualize the heat transfer process, we use `mesh` to plot a 3D graph:



The temperature can be shown more intuitively using `contourf`:



The `script` to reproduce these results along with `heatrans.m` bundled with this report and this document itself are all licensed under a Creative Commons Attribution-ShareAlike 4.0 International License.